# VARIOUS REGULARIZATION FUNCTIONS IN SYSTEM IDENTIFICATION PROBLEMS FOR SOLIDS

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**ABSTRACT**

This paper presents various regularization functions, which are employed to overcome instabilities of system identification problems. Since a regularization function define the solution space of a given problem, it should well represent both mathematical and physical characteristics of the original problem. The regularization function can be derived from the integrability condition of the solution in conjunction with physical consideration. The $L_1$-norm as well as $L_2$-norm of system parameters is used to define the regularity functions. Various types of regularization functions and their characteristics are discussed.

## INTRODUCTION

System identification (SI) algorithms have been widely used for the last few decades in the area of structural engineering to identify mechanical systems and to detect damage in structures. However, SI algorithms based on the minimization of the least squared error between measured and computed responses suffer from inherent instabilities caused by the ill-posedness of inverse problems [1,2]. The instabilities are characterized by the non-uniqueness and discontinuity of solutions. In particular, when measured data are polluted with noise or when a finite element model used for the SI does not represent actual situations, the instabilities become very severe.

To overcome the instabilities of inverse problems, the regularization technique has been utilized [3]. In a regularization technique, a regularization function is introduced as additional constraints to define the solution space of a system identification problem. Therefore, the regularization function should be selected so that physical and mathematical characteristics of the given problem can be properly represented. This paper reviews various types of regularization functions that have been successfully applied to structural system identification problems. Both the $L_1$-norm [4] and the $L_2$-norm are utilized to define regularity conditions of solutions of SI problems. Two different regularization techniques – Tikhonov regularization scheme and truncated singular value decomposition – are considered to impose the regularization functions.

## SYSTEM IDENTIFICATION PROBLEMS

The system properties of a solid are estimated in SI by minimizing the least squared errors between measured and calculated responses under known input conditions.

$$\underset{\mathbf{X}}{\text{Minimize}}\, \Pi_E = \frac{1}{2}\sum_{i=1}^{nlc}\left\|\tilde{\mathbf{u}}_i(\mathbf{X}) - \bar{\mathbf{u}}_i\right\|_2^2 \quad \text{subject to} \;\; \mathbf{R}(\mathbf{X}) \le 0 \qquad (1)$$

Here, $\tilde{\mathbf{u}}_i$, $\bar{\mathbf{u}}_i$, **X**, **R** and *nlc* are calculated responses by the mathematical model, measured responses at observation points for input case *i*, a system parameter vector, a constraint vector for the system parameters and the number of load cases, respectively, while $\|\cdot\|_2$ denotes the 2-norm of a vector. The system parameter vector **X** represents the discretized system properties. The responses of a solid are calculated by using the FEM or similar discretization methods.

$$\mathbf{K}(\mathbf{X})\mathbf{u}_i = \mathbf{P}_i \quad \text{for} \quad i = 1, \cdots, nlc \tag{2}$$

where $\mathbf{P}_i$ is the nodal input vector for input case *i*.

The SI problems defined by the minimization problem (1) exhibit strong instabilities characterized by the discontinuity and non-uniqueness of solutions. The instabilities become severe when the measured responses contain noise and/or the number of measured responses are smaller than that of the degrees of freedom in the mathematical model. To avoid the instabilities of SI problems, a proper solution space for a SI problem should be supplied along with the minimization problem (1).

## REGULARIZATION FUNCTIONS

The solution space of the optimization problem (1) is defined by a regularization function. The $L_2$-norm is widely employed in various engineering problems to represent piecewise continuous system parameters.

$$\Pi_R = \|x - x_0\|^2_{L_2(V)} = \int_V (x - x_0)^2 \, dV < \infty \tag{3}$$

where $\Pi_R$, $x$ and $x_0$ denote the regularization function, the system parameter and its a prior information, respectively, and $V$ represents the structural domain. The $L_2$-norm based regularization functions effectively stabilize ill-posedness of SI. However, they may produce smeared solutions due to smoothing characteristics of 2-norm minimization in case the actual distribution of the system parameters being estimated is discontinuous in a given domain. To overcome the smearing effect of the $L_2$-norm, and represent the discontinuity of system parameters more accurately, the $L_1$-norm of the system parameter is often utilized.

$$\Pi_R = \|x - x_0\|_{L_1(V)} = \int_V |x - x_0| \, dV < \infty \tag{4}$$

The regularity condition of a piecewise continuous function can be given alternatively by the $L_1$-norm of the gradient of the system parameter parameters.

$$\|\nabla(x - x_0)\|_{L_1(V)} = \int_V |\nabla(x - x_0)| \, dV < \infty \tag{5}$$

where $\nabla$ is gradient operator. In case the domain of a structure is discretized by finite elements and the system parameters are constant within elements, the $L_1$-regularity condition (5) is discretized as follows.

$$\int_V |\nabla(x-x_0)| dV \approx \sum_e \int_{V_e} |\nabla(x-x_0)| dV + \sum_{k=1}^{n_B} \left|(X_k^1 - X_k^2) - ((X_k^2)_0 - (X_k^2)_0)\right| l_k$$
$$= \sum_{k=1}^{n_B} \left|(X_k^1 - (X_k^2)_0) - (X_k^2 - (X_k^2)_0)\right| l_k \tag{6}$$

where $n_B$, and $l_k$ are the number of inter-element boundaries and the length of the $k$-th inter-element boundary, respectively, while $X_k^1$ and $X_k^2$ are system parameters of two elements sharing the $k$-th inter-element boundary. Since the system property is assumed to be constant in an element, the domain integral in (6) vanishes. The second term of (6) represents jumps of the system parameters across inter-element boundaries.

Lee et al [3] applied a SI algorithm to identify a boundary curve of an inclusion in a finite body. Since the boundary curve of an inclusion is continuous in the curve parameter space, the proper solution space of a boundary curve can be defined as

$$\int_\Gamma ((\frac{dC_x}{ds})^2 + (\frac{dC_y}{ds})^2) ds = \sum_{e=1}^{nelem} \int_{t_e} ((\frac{dC_x}{ds})^2 + (\frac{dC_y}{ds})^2) ds < \infty \tag{7}$$

Here, $C_x$, $C_y$, $\Gamma$, and $s$ are $x$-component, $y$-component of the boundary curve, the boundary curve and the curve parameter, respectively, and *nelem* is the number of elements in the discretized boundary curve.

For the system identification in time domain, two types of regularization functions have been proposed by Lee et al [5,6]. In case system parameters are continuous in time, the first-order time derivative of the system parameter should be piecewise continuous, and thus the following regularization function based on the $L_2$-norm is appropriate [5].

$$\Pi_R = \int_0^\tau \left\| \frac{d\mathbf{x}(t)}{dt} \right\|^2 dt < \infty \tag{8}$$

where $t$ denotes time. When the system parameter changes abruptly or piecewise continuous fashion in time, the regularization function may be defined by the $L_2$-norm of the system parameters as follows.

$$\Pi_R = \int_0^\tau \|\mathbf{x}(t)\|^2 dt < \infty \tag{9}$$

To avoid the smearing effect of the $L_2$-norm defined in (9), the $L_1$-norm of the first order time derivative may be employed [6].

$$\Pi_R = \int_0^\tau \left| \frac{d\mathbf{x}(t)}{dt} \right| dt < \infty \tag{10}$$

## IMPOSITION OF REGULARITY CONDITIONS

The regularity conditions presented in the previous section are imposed to the original minimization problem (1) by the regularization techniques, among which the Tikhonov regularization technique and the truncated singular value deposition (TSVD) are widely used [2]. In the Tikhonov regularization technique, the regularity condition is added to the original error function, and the optimization is performed for the regularized error function as follows.

$$\underset{\mathbf{X}}{\text{Minimize}}\ \Pi = \frac{1}{2}\left\|\tilde{\mathbf{U}}(\mathbf{X}) - \overline{\mathbf{U}}\right\|_2^2 + \lambda\Pi_R \text{ subject to } \mathbf{R}(\mathbf{X}) \le 0 \tag{11}$$

where $\lambda$ is the regularization factor, which adjusts the degree of regularization, and $\Pi_R$ represents a proper regularity conditions. The minimization problem defined in (11) is nonlinear with respect to the system parameters. However, a Newton type algorithm, which requires the gradient information of $\Pi$, cannot be applied to solve (11) in case a non-differentiable 1–norm is adopted for the regularization function.

For the SI with a non-differentiable $L_1$-regularization function, the TSVD is utilized to impose the regularity condition iteratively in the optimization of the error function. In this method, the incremental solution of the error function is obtained by solving the quadratic sub-problems without the constraints. The noise-polluted solution components are truncated from the incremental solution. Finally, the regularity condition is imposed to restore the truncated solution components and the constraints. The above procedure is defined as follows.

$$\underset{\mathbf{X}}{\text{Minimize}}\ \Pi_R \text{ subject to } \mathbf{R}(\mathbf{X}) \le 0 \text{ and } \underset{\mathbf{X}}{\text{Minimize}}\ \Pi_E = \frac{1}{2}\left\|\tilde{\mathbf{U}}(\mathbf{X}) - \overline{\mathbf{U}}\right\|_2^2 \tag{12}$$

Detailed solution procedures of the TSVD are found in references [2] and [4].

## CONCLUSIONS

Various regularization functions, which are used to alleviate the ill-posedness of inverse problems in solids, are reviewed. Each regularization function is derived so that it represents proper integrability conditions of system parameters of a given problem. To obtain a meaningful solution of a given SI problem, a proper regularization function should be selected.

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