공학박사학위논문

# Acceleration-Energy Filter and Bias Compensation for Stabilizing Equation Error Estimator in Inverse Analysis Using Dynamic Displacement

동적 변위를 사용하는 역해석 문제에서 Equation Error Estimator 안정화를 위한 가속도-에너지 필터 및 편향성 보정 기법

2015년 8월

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### Abstract

New stabilization schemes which correct the equation error estimator (EEE) in the inverse analysis using dynamic responses of linear elastic continua are presented. The goal of the inverse analysis run in these cases is the proper identification of material properties. Stabilization schemes consist of the acceleration-energy filter and the bias compensation.

The acceleration-energy filter stabilizes the ill-posedness of the inverse analysis. The acceleration-energy filter replaces the techniques known as truncated singular value decomposition (TSVD),  $L_1$ -norm regularization and  $L_2$ -norm regularization (or Tikhonov regularization). Existing regularization techniques do not work properly for cases involving hard inclusions, i.e., tumors of organ and suspensions of vehicles. The Acceleration-energy filter, however, work properly for cases involving both hard and soft inclusions. The acceleration-energy filter is separated into the acceleration filter and the energy filter. Dividing them in this manner simplifies a filtering process.

The acceleration filter imposes finiteness condition of accelerations, the second derivatives of the measured displacements. Accelerations can be considered as finite functions when impact loads do not exist. The acceleration filter requires two initial and two final values, but the overlapping moving time window technique is employed so that the initial and final values can be ignored.

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The final form of the acceleration filter is a low-pass finite impulse response (FIR) filter. However, the acceleration filter differs from typical low-pass FIR filters because it has physical meaning which guarantees consistency with the energy regularization.

The energy filter imposes finiteness of strain energy, which is internal energy of linear elastic continua. The final form of the energy filter is very similar to low-pass spatial filters used with image processing, but the energy filter has three advantages. The first of these are the boundary conditions. The boundary conditions of the energy filter are identical to these of an equilibrium equation for the continuum, and are always satisfied by all continuum examples. The second is the available meshes. The energy filter involves the connectivity information of nodes and can handle complicatedly meshed FEM models, whereas typical lowpass spatial filters can handle only rectangular meshes. The third advantage is the physical meaning which guarantees consistency with the acceleration filter.

The acceleration filter and the energy filter must satisfy consistency of the elastic waves and the temporal wave. The solution of the inverse analysis without the consistency is not trustable because the strain and the acceleration do not have equivalent information. The physical meaning of two filters gives consistency between two filter.

The biases of the solutions are ignored in existing studies. However, the inverse analysis using EEE for linear elastic continua must consider the biases. If the noise variances are known, the biases of the solution could be perfectly

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eliminated by means of bias compensation.

Aluminum plate and medical imaging examples are demonstrated to show the effectiveness of the schemes described above.

Key words : Equation Error Estimation (EEE), System Identification,Inverse Analysis, Identifying Material Properties, Acceleration-Energy Filter,Temporal-Spatial Filter, Bias-compensation, Linear Elastic Continua,Medical Imaging

Student Number: 2010-30239

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## **1. Introduction**

Currently, system identification (SI) is an important issue in many fields, including the civil engineering, mechanics, signal processing, shipbuilding, medical engineering and other fields. SI reconstructs material properties and provides important information for the maintenance and safety of structures. SI is also known as inverse analysis.

Inverse analysis is known to be able to identify unknown system parameters and to reconstruct inputs for producing desired outputs. The inverse analysis for unknown system parameters detects damage in engineering structures and/or tumors in human bodies. The identified stiffness, mass or damping can take the form of damage indexes. The inverse analysis for the reconstruction of the input is used for control and signal processing, but these areas are outside the scope of this thesis.

Error minimization schemes are employed to define the inverse analysis. Two types of error estimators, output error estimation (OEE) and equation error estimation (EEE), are typically used. EEE estimates errors in the governing equation or the weak form of it. The inverse analysis using EEE is suitable for real-time processes because it is a quadratic optimization problem and a parallelizable algorithm when the governing equation of the structure is linear [Hjelmstad 1995]. Medical imaging, feedback control and structural safety

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management require the real-time inverse analysis. OEE estimates errors between responses from unknown models and measured responses. The inverse analysis using OEE is an iterative scheme and not a parallelizable algorithm [Hjelmstad 1995, Banan 1995, Huang 2001]. Since quadratic problems have a lot of advantages, EEE is utilized in this thesis.

The target structures are described by the linear elastic continua because the continuum explain mosts of engineering solid structures. To solve the inverse analysis numerically, the target continuum must be discretized by a numerical model. In this thesis, the finite element method (FEM) is used as a numerical model. The FEM is the best model with regard to forward analysis for continua; however, the FEM amplifies noise in measurement when it is employed to solve the inverse analysis. The relationship between strain and displacement amplifies the noise.

The inverse analysis using minimization schemes leads to ill-posedness when modeling errors and measurements noise exist [Bui 1994, Hansen 1998]. The regularization has been proposed to reduce ill-posedness [Vogel 1986, Park 2007, Lee 1999, Park 2001]. The regularization schemes work well for cases involving soft inclusions. However, these efforts do not apply to cases involving hard inclusions. Soft inclusion cases are the most common type of engineering damage, but there are a number of hard inclusion cases, such as impurities of concrete structures, tumors in human bodies and suspension stiffening of vehicles. The acceleration-energy filter is proposed to stabilize the ill-posedness which arises in cases involving both soft and hard inclusions. It imposes the physical laws which the measurements must satisfy. The measurements must satisfy two laws of the continuum, the finiteness of the strain energy and acceleration. The accelerationenergy filter is separated into the acceleration filter and the energy filter. Dividing them in this manner simplifies a filtering process. Both filters function as lowpass filters to suppress noise amplification from differentiations. The effects of the acceleration filter are similar to those of existing noise suppression filters. However, the physical meaning of the acceleration filter is related to the physical meaning of the energy filter, thus provides consistency between the two filters. The energy filter is similar to low-pass spatial filters, which is used for image processing. However, the modeling information of the target continuum is included in the energy filter. While image filters consider only nodes arranged in a rectangular, the modeling information allows for the energy filter to consider nodes in more complicated connection. Moreover, the energy filter can consider nodes around boundaries properly, as the boundary condition used in the energy filter is identical to that of the continuum. The acceleration filter and the energy filter must satisfy consistency of the elastic waves and the temporal wave. The solution of the inverse analysis without the consistency is not trustable because the strain and the acceleration do not have equivalent information. The physical meaning of two filters gives consistency between two filter.

Solutions of the inverse analysis using EEE have not only ill-posedness but also bias in the solution. When the measurement is polluted by noise, the EEE

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includes squared error terms, which are directly proportional to the variance of the noise. In most inverse analyses using EEE, the bias is negligible or easy to remove [Nguyen 1993, Hjelmstad 1995, Haykin 2008, Ikenoue 2009, Zhang 2011]. However, the bias inherent in the inverse analysis using EEE for the linear elastic continua cannot be ignored, as the noise is amplified twice during the inverse analysis process and is complexly involved in the FEM model. The squared error terms lead to fixed direction errors of the solutions, i.e., the biases, and cannot be eliminated by an infinite measuring time. The bias compensation can reconstruct the bias terms and remove the biases of the solutions when the variance of the noise is known. A case of unknown variance is also introduced in this thesis.

A time-domain analysis using dynamic displacement is employed rather than a frequency-domain analysis. A time-domain analysis is more sensitive to local inclusion than a frequency-domain analysis. A frequency-domain analysis has advantages which simplify problems, but is not sensitive to local inclusion [Raghavendrachar 1992]. Laser displacement measures are commonly used [Park 2013, Choi 2013]. This equipment is more expensive than accelerometers and requires reference points. However, laser displacement measures are more accurate than accelerometers. Moreover, LIDAR, a laser scanning technique, provides a high spatial resolution. Recently, there have been numerous studies which have attempted to reduce cost of the displacement measure. Several studies have focused on a vision-based displacement measurement technique [Kim 2012, Kim 2013a, Kim 2013b]. It requires only a small number of video cameras to

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measure the displacement of large structures. A light emitting diode (LED) can be applied to improve the vision-based displacement measurement technique [Wahbeh 2003]. Kinect, an input device developed by Microsoft as part of their XBOX 360, is applied as displacement measure to reduce costs [Qi 2014]. Specifically, a ultrasonic elastography device provides displacement measurements [Bercoff 2003, Bercoff 2004, Park 2006, Park 2009]. Ultrasonic elastography devices incur low costs and provide high-quality information for diagnosis.

An example and an application are introduced to assess the accelerationenergy filter and the bias compensation. The first example is an aluminum plate which is under a plane stress condition. It shows how the inverse analysis using EEE and stabilization schemes work. The second example is a human skin tissue. It uses the Helmholtz equation, which is an equation for incompressible continuum. The inverse analysis and stabilization schemes are slightly different from those of the inverse analysis for the original continuum. The example shows an algorithm for medical imaging using ultrasonic equipment.

# **2.** Equation Error Estimator in Inverse Analysis Using Dynamic Displacement

### 2.1 Definition of Inverse Problem Using Minimization

There are numerous engineering demands to identify unknown system parameters of continua, especially stiffness parameters. These identifying schemes are known as system identification (SI) and/or inverse analysis. Inverse analysis typically uses various types of measurements, including the displacement, velocity, acceleration, and strain. With regard to the types of measurements, displacement includes the lowest frequency information. It works well with massive structures such as civil structures, aircraft, and ships. For wave problems such as those in ultrasonic medical imaging, only displacement is valid because it includes the information pertaining to frequencies around 0Hz. Acceleration and velocity do not include this type of information.

Unknown stiffness parameters are estimated by inverse analysis for linear elastic continua. The governing equation of the continuum is a function of the displacement, system parameters and external forces.

$$C_{ijkl} \frac{\partial^2 u_k}{\partial x_j \partial x_l} + b_i - \rho \frac{\partial^2 u_i}{\partial t^2} = 0$$
(2.1)

where  $C_{iikl}$ ,  $x_i$ ,  $u_i$ , t,  $b_i$ ,  $\rho$  and  $\partial$  are the elasticity tensor, the Cartesian

tensor, the displacement tensor, the time dimension, the body force tensor, the mass density and the symbol of partial differentiation, respectively. The displacement can be calculated via the system parameters and the external forces.

Error minimization schemes are employed for identifying the stiffness parameters, which are parts of the system parameters. Output error estimation (OEE) and equation error estimation (EEE) are typical error estimators.

OEE estimates errors between responses from unknown models and measured responses. The next equation represents inverse analysis using OEE which uses the  $L_2$ -norm [Hjelmstad 1996, Ge 1998, Huang 2001, Kang 2005].

$$\min \Pi = \int_{T} \int_{V_t} \left\| u(C_{ijkl}) - \overline{u} \right\|_2^2 dV dt$$
(2.2)

Here, T,  $V_t$ ,  $C_{ijkl}$ , u and  $\overline{u}$  are the total measuring time, the whole domain of the continuum, the unknown stiffness parameter, the calculated displacement from the governing equation (or from a numerical model of the equation) and the measured displacement, respectively. Fig. (2.1) shows the basic concept of inverse analysis using OEE. Eq. (2.2) is a non-linear minimization problem because  $u(C_{ijkl})$  is calculated by solving forward problems, including a matrix inversion calculation. Accordingly, inverse analysis using OEE requires a considerable amount of computational effort and is not suitable for real-time SI.



However, responses of all degrees of freedom (DOF) do not have to be measured. This is one advantage for selecting measurement points. Also, inverse analysis using OEE does not have biases in its solutions.

EEE estimates errors in the governing equation or the weak form of it. The inverse analysis using EEE with the  $L_2$ -norm is in the next.

$$\min \Pi = \int_{t} \int_{V_{t}} \left\| C_{ijkl} \frac{\partial^{2} \overline{u}_{k}}{\partial x_{j} \partial x_{l}} - \rho \frac{\partial^{2} \overline{u}_{i}}{\partial t^{2}} \right\|_{2}^{2} dV dt$$
(2.3)

In this equation, T,  $V_t$ ,  $C_{ijkl}$ , u and  $\rho$  are the total measurement time, the whole domain of the continuum, the unknown stiffness parameters, the measured displacements and the (known) mass densities, respectively. Fig. (2.2) shows the basic concept of inverse analysis using EEE. Minimization problems using EEE are quadratic problems when the equations are linear with respect to unknowns. The quadratic problem requires a short and predictable computational time.



Figure 2.2 Concept of EEE

Inverse analysis of linear elastic continua using EEE is a parallelizable algorithm, because each time step and each of the terms are perfectly independent from each other. The quadratic object function and the parallelizable algorithm make inverse analysis using EEE suitable for real-time or semi-real-time processes. However, the solutions from the inverse analysis using EEE contain biases. The biases must be handled by a bias compensation scheme, as introduced in section 2.4. Another disadvantage is that all displacements, velocities and accelerations on the all DOF are required. If only one of these responses is measured, the other responses must be derived by differentiation or integration. The noise sources in the measurement are amplified during the differentiation and integration process. The amplified noise amplifies the ill-posedness and the bias of the solutions. This amplification must be handled by the acceleration-energy filter, which is introduced in chapter 3.

As introduced in chapter 1, inverse analysis using EEE is feasible for real-time processes and thus the acceleration-energy filter in this paper focus on EEE. The filter here can be applied to inverse analysis using OEE, but this is outside the scope of this thesis.

Several assumptions are necessary to simplify the problems. The first is that the external forces and the body forces are ignored because forces from outside of the continua are difficult to measure. Transient loads, which are released immediately before the measurement processes start, satisfy the first assumption. When this assumption is valid, the body force term  $b_i$  in Eq. (2.1) becomes zero. A second assumption is that damping can be ignored. A short measurement time makes damping ignorable. This assumption is satisfied when relatively few vibrations are included in the measurement time. A third assumption is that the mass density and the Poisson ratio are known. The final assumption here is the assumption of isotropy continuum.

### 2.2 FEM for Discretizing EEE

Eq. (2.3) must be discretized, as measured displacements are usually discretized in both temporal and spatial dimensions. The finite element method (FEM), the rectangle method and the central finite difference method (FDM) are utilized for discretizing the spatial domain, the temporal integration and the temporal differentiation, respectively.

Originally, Eq. (2.3) is discretized but cannot be. The weak form of the governing equation is employed for replacing the governing equation in the Eq. (2.3). Spatial discretization starts with the variation of Eq. (2.1), without the body force  $b_i$ .

$$\int_{V} \delta \overline{u}_{i} C_{ijkl} \frac{\partial^{2} \overline{u}_{k}}{\partial x_{j} \partial x_{l}} dV - \int_{V} \delta \overline{u}_{i} \rho \frac{\partial^{2} \overline{u}_{i}}{\partial t^{2}} dV$$
(2.4)

Here,  $\overline{u}_i$  is the measured displacement. The equation above is integrated by parts and discretized by FEM.

$$\int_{V} \delta \overline{u}_{i} \rho \frac{\partial^{2} \overline{u}_{i}}{\partial t^{2}} dV + \int_{V} \frac{\partial \delta \overline{u}_{i}}{\partial x_{j}} C_{ijkl} \frac{\partial \overline{u}_{k}}{\partial x_{l}} dV - \int_{s} \delta \overline{u}_{i} C_{ijkl} \frac{\partial \overline{u}_{k}}{\partial x_{l}} n_{j} dS$$
(2.5a)

$$\approx \delta \overline{\mathbf{U}} \left[ \sum_{e} \left( \int_{V^{e}} \rho^{e} \mathbf{N}^{T} \mathbf{N} dV \right) \overline{\mathbf{U}} + \sum_{e} E^{e} \left( \int_{V^{e}} \mathbf{B}^{T} \mathbf{D} \mathbf{B} dV \right) \overline{\mathbf{U}} \right] = 0$$
(2.5b)

where **N**, **B**, **D**,  $\overline{\mathbf{U}}$  and  $E^e$  are the shape function matrix, the first derivatives of the shape function matrix, the constitutive matrix with the unit Young's modulus, the measured displacement vector and the Young's modulus of each element, respectively. The superscript *e* refers to each element.  $\sum_{e}$  () represents a structural compatibility summation. The last term in Eq. (2.5a) is a boundary condition and is eliminated on the fixed and free boundaries. The sufficient condition of Eq. (2.5b) is the next.

$$\sum_{e} \left( \int_{V^{e}} \rho^{e} \mathbf{N}^{T} \mathbf{N} dV \right) \overline{\mathbf{U}} + \sum_{e} E^{e} \left( \int_{V^{e}} \mathbf{B}^{T} \mathbf{D} \mathbf{B} dV \right) \overline{\mathbf{U}}$$
(2.6a)

$$=\sum_{e}\mathbf{m}^{e}\mathbf{L}_{2}\overline{\mathbf{U}}+\sum_{e}E^{e}\mathbf{q}^{e}\overline{\mathbf{U}}=0$$
(2.6b)

 $\mathbf{m}^{e}$  and  $\mathbf{q}^{e}$  are defined below

$$\mathbf{m}^{e} = \int_{V^{e}} \rho \mathbf{N}^{T} \mathbf{N} dV$$
(2.7a)

$$\mathbf{q}^{e} = \int_{V^{e}} \mathbf{B}^{T} \mathbf{D} \mathbf{B} dV$$
(2.7b)

Since solutions of Eq. (2.6b) are the optimal solutions of the equilibrium equation (Eq. 2.1), Eq. (2.6b) replaces the equilibrium equation in Eq. (2.3).

$$\min \Pi = \int_{t} \left\| \sum_{e} \mathbf{m}^{e} \overline{\mathbf{U}} + \sum_{e} E^{e} \mathbf{q}^{e} \overline{\mathbf{U}} \right\|_{2}^{2} dt$$
(2.8)

The rectangle method and the central FDM are employed for descretizing the temporal integration and the displacement-acceleration relationship, respectively.

$$\min \Pi \approx \sum_{t} \left\| \sum_{e} \mathbf{m}^{e} \overline{\mathbf{U}}_{t} + \sum_{e} E^{e} \mathbf{q}^{e} \overline{\mathbf{U}}_{t} \right\|_{2}^{2} \Delta t$$
(2.9a)

$$\approx \sum_{t} \left\| \sum_{e} \mathbf{m}^{e} \mathbf{L}_{2} \overline{\mathbf{U}}_{t} + \sum_{e} E^{e} \mathbf{q}^{e} \overline{\mathbf{U}}_{t} \right\|_{2}^{2} \Delta t$$
(2.9b)

The subscript t indicates each time step. The central FDM matrix for second order derivatives using second order accuracy  $\mathbf{L}_2$  is defined below.

Since  $\Delta t$  in Eq. (2.9b) does not effect to result of the minimization problem, the discretized object function of inverse analysis using EEE is given below.

$$\min \Pi = \sum_{t} \left\| \sum_{e} \mathbf{m}^{e} \mathbf{L}_{2} \overline{\mathbf{U}}_{t} + \sum_{e} E^{e} \mathbf{q}^{e} \overline{\mathbf{U}}_{t} \right\|_{2}^{2}$$
(2.11)

The solution of the quadratic minimization problem is given below.

$$\frac{\partial \Pi}{\partial E^{i}} = \sum_{t} \left[ \overline{\mathbf{U}}_{t}^{T} \sum_{e} (\mathbf{q}^{i^{T}} \mathbf{m}^{e} \mathbf{L}_{2}) \overline{\mathbf{U}}_{t} \right] + \sum_{t} \left[ \overline{\mathbf{U}}_{t}^{T} \sum_{e} (E^{e} \mathbf{q}^{i^{T}} \mathbf{q}^{e}) \overline{\mathbf{U}}_{t} \right] = 0$$
(2.12)

The number of these equations is identical to the number of elements. A matrix form of the equation above is in the next.

$$\mathbf{M} + \mathbf{Q}\mathbf{E} = \mathbf{0} \tag{2.13a}$$

$$\mathbf{E} = -\mathbf{Q}^{-1}\mathbf{M} \tag{2.13b}$$

 $\mathbf{M}, \ \mathbf{Q}$  and  $\mathbf{E}$  are defined below.

$$\mathbf{M} = \sum_{t=1}^{n} \begin{bmatrix} \overline{\mathbf{U}}_{t}^{T} \mathbf{q}^{1^{T}} (\mathbf{m}^{1} + \mathbf{m}^{2} + \dots + \mathbf{m}^{e}) \mathbf{L}_{2} \overline{\mathbf{U}}_{t} \\ \overline{\mathbf{U}}_{t}^{T} \mathbf{q}^{2^{T}} (\mathbf{m}^{1} + \mathbf{m}^{2} + \dots + \mathbf{m}^{e}) \mathbf{L}_{2} \overline{\mathbf{U}}_{t} \\ \vdots \\ \overline{\mathbf{U}}_{t}^{T} \mathbf{q}^{e^{T}} (\mathbf{m}^{1} + \mathbf{m}^{2} + \dots + \mathbf{m}^{e}) \mathbf{L}_{2} \overline{\mathbf{U}}_{t} \end{bmatrix}$$
(2.14a)  
$$\mathbf{Q} = \sum_{t=1}^{n} \begin{bmatrix} \overline{\mathbf{U}}_{t}^{T} \mathbf{q}^{1^{T}} \overline{\mathbf{q}}^{1} \overline{\mathbf{U}}_{t} & \overline{\mathbf{U}}_{t}^{T} \mathbf{q}^{1^{T}} \mathbf{q}^{2} \overline{\mathbf{U}}_{t} & \dots & \overline{\mathbf{U}}_{t}^{T} \mathbf{q}^{1^{T}} \mathbf{q}^{e} \overline{\mathbf{U}}_{t} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{\mathbf{U}}_{t}^{T} \mathbf{q}^{e^{T}} \mathbf{q}^{1} \overline{\mathbf{U}}_{t} & \overline{\mathbf{U}}_{t}^{T} \mathbf{q}^{2^{T}} \mathbf{q}^{2} \overline{\mathbf{U}}_{t} & \dots & \overline{\mathbf{U}}_{t}^{T} \mathbf{q}^{2^{T}} \mathbf{q}^{e} \overline{\mathbf{U}}_{t} \end{bmatrix}$$
(2.14b)  
$$\mathbf{E} = \begin{bmatrix} E^{1} \\ E^{2} \\ \vdots \\ E^{e} \end{bmatrix}$$
(2.14c)

It is important to note that the measured displacement vector  $\overline{\mathbf{U}}$  is squared. The squared terms cause bias in the solutions. This is discussed in the next section.

### 2.3 Effect of Noise in Displacement

#### 2.3.1 Noise Amplification by Differentiation

The measured displacement always includes noise. The noise is amplified by displacement-acceleration and displacement-strain relationships, which are the second and the first order differentiations, respectively. To understand the noise amplification of differentiations, a frequency-domain analysis is useful. The Fourier transform and the transfer function provide information on the frequency domain. [Bracewell 2000]

The Fourier transform (Eq. 2.15) provides frequency domain information from a function or signal of the original domain.

$$\mathscr{F}[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx = \hat{f}(\omega)$$
(2.15)

Here,  $\mathcal{F}$ , f(x) and  $\hat{f}(\omega)$  denote the Fourier transform, a function/signal of the original domain and a function/signal of the frequency domain, respectively. The original domain is a temporal or spatial domain in this paper, and the frequency domain is a temporal or spatial frequency domain as well.

A function of the frequency domain is a transfer function. The definition if the transfer function is the linear mapping of the input and the output in the frequency domain.

$$Y(\omega) = H(\omega)X(\omega)$$
(2.16)

Where  $\omega$ ,  $H(\omega)$ ,  $Y(\omega)$  and  $X(\omega)$  are the frequency in radian, the transfer function, the Fourier transform of the output and the Fourier Transform of the input, respectively. The Fourier transform of the differentiation is the next equation.

$$\mathscr{F}\left[\frac{d^n f(x)}{dx^n}\right] = (\mathrm{i}\omega)^n \,\hat{f}(\omega) \tag{2.17}$$

In this equation, i and *n* are the unit of a imaginary number and the order of the differentiation. The transfer function  $H(\omega) = (i\omega)^n$  is a monotonically increasing function and indicates that the high frequency signal is amplified by the differentiation. The amplified high-frequency noise, especially near the Nyquist frequency, act as a series of Dirac delta functions. Fig. (2.3) shows the amplification levels of second differentiation at a normalized sampling rate (original sampling rate =  $10^5$  Hz; for the Nyquist frequency =  $5 \times 10^4$ Hz; normalized by radian). Amplification levels of the second FDM are also displayed in the same graph. Even if the noise amplification of the FDM is lower than the



Figure 2.3 Transfer function of the second differentiation (absolute value)

amplification of exact differentiation, it remains critical. Figs. (2.4a, b and c) show the noise-free and noisy differentiation of the harmonic function combination. Fig. (2.4a) is displacement  $u(t) = \sin(300 \times 2\pi t) + \cos(250 \times 2\pi t)$  and the noised displacement. Fig. (2.4b and c) are the accelerations, the second derivatives of the exact/noisy displacements and the large scale of (b), respectively. As shown in the Fig. (2.4b), the amplified high-frequency noise acts as a series of Dirac delta functions. This breaks physical laws, the finiteness of the strain energy and acceleration, for the dynamic response of the continuum and causes error in the solutions of the inverse analyses.



Figure 2.4 Noise free and noised differentiation of harmonic function combination (a)original harmonic function (b)second derivatives (c)large scale of derivatives

### 2.3.2 Decomposition of Noise Effects

The amplified noise in the measured displacement effects to the solutions of EEE. To analyze effects of noise in the measured displacements,  $\overline{\mathbf{U}}_t$  in Eqs. (2.14a and b) is replaced by  $(\overline{\mathbf{U}}_e + \overline{\mathbf{U}}_n)$ .  $\overline{\mathbf{U}}_e$  and  $\overline{\mathbf{U}}_n$  represent the exact displacement and the noise in the measured displacement, respectively.

$$\begin{aligned} \mathbf{Q} &= \sum_{t=1}^{n} \begin{bmatrix} \overline{\mathbf{U}}_{t}^{T} \mathbf{q}^{1} \overline{\mathbf{U}}_{t} & \cdots & \overline{\mathbf{U}}_{t}^{T} \mathbf{q}^{1}^{T} \mathbf{q}^{e} \overline{\mathbf{U}}_{t} \\ \vdots & \ddots & \vdots \\ \overline{\mathbf{U}}_{t}^{T} \mathbf{q}^{e^{T}} \mathbf{q}^{1} \overline{\mathbf{U}}_{t} & \cdots & \overline{\mathbf{U}}_{t}^{T} \mathbf{q}^{e^{T}} \mathbf{q}^{e} \overline{\mathbf{U}}_{t} \end{bmatrix} \\ &= \sum_{t=1}^{n} \begin{bmatrix} \overline{\mathbf{U}}_{e}^{T} \mathbf{q}^{1} \overline{\mathbf{U}}_{e} & \cdots & \overline{\mathbf{U}}_{e}^{T} \mathbf{q}^{1}^{T} \mathbf{q}^{e} \overline{\mathbf{U}}_{e} \\ \vdots & \ddots & \vdots \\ \overline{\mathbf{U}}_{e}^{T} \mathbf{q}^{e^{T}} \mathbf{q}^{1} \overline{\mathbf{U}}_{e} & \cdots & \overline{\mathbf{U}}_{e}^{T} \mathbf{q}^{e^{T}} \mathbf{q}^{e} \overline{\mathbf{U}}_{e} \end{bmatrix} \\ &+ \sum_{t=1}^{n} \begin{bmatrix} \overline{\mathbf{U}}_{e}^{T} \mathbf{q}^{1} \overline{\mathbf{U}}_{e} & \cdots & \overline{\mathbf{U}}_{e}^{T} \mathbf{q}^{e^{T}} \mathbf{q}^{e} \overline{\mathbf{U}}_{e} \\ \vdots & \ddots & \vdots \\ \overline{\mathbf{U}}_{e}^{T} \mathbf{q}^{e^{T}} \mathbf{q}^{1} \overline{\mathbf{U}}_{n} & \cdots & \overline{\mathbf{U}}_{n}^{T} \mathbf{q}^{e^{T}} \mathbf{q}^{e} \overline{\mathbf{U}}_{n} \end{bmatrix} \\ &+ \sum_{t=1}^{n} \begin{bmatrix} \overline{\mathbf{U}}_{e}^{T} \mathbf{q}^{1} \overline{\mathbf{U}}_{n} + \overline{\mathbf{U}}_{n}^{T} \mathbf{q}^{1} \overline{\mathbf{U}}_{e} & \cdots & \overline{\mathbf{U}}_{e}^{T} \mathbf{q}^{e^{T}} \overline{\mathbf{U}}_{n} \\ \vdots & \ddots & \vdots \\ \overline{\mathbf{U}}_{e}^{T} \mathbf{q}^{e^{T}} \mathbf{q}^{1} \overline{\mathbf{U}}_{n} + \overline{\mathbf{U}}_{n}^{T} \mathbf{q}^{1} \overline{\mathbf{U}}_{e} & \cdots & \overline{\mathbf{U}}_{e}^{T} \mathbf{q}^{1^{T}} \mathbf{q}^{e} \overline{\mathbf{U}}_{n} + \overline{\mathbf{U}}_{n}^{T} \mathbf{q}^{e^{T}} \mathbf{q}^{e} \end{bmatrix} \\ &+ \sum_{t=1}^{n} \begin{bmatrix} \overline{\mathbf{U}}_{e}^{T} \mathbf{q}^{1} \overline{\mathbf{U}}_{n} + \overline{\mathbf{U}}_{n}^{T} \mathbf{q}^{1^{T}} \mathbf{q}^{1} \overline{\mathbf{U}}_{e} & \cdots & \overline{\mathbf{U}}_{e}^{T} \mathbf{q}^{e^{T}} \mathbf{q}^{e} \overline{\mathbf{U}}_{n} + \overline{\mathbf{U}}_{n}^{T} \mathbf{q}^{e^{T}} \mathbf{Q}_{e} \end{bmatrix} \\ &= \mathbf{Q}_{\text{exact}} + \mathbf{Q}_{\text{bias}} + \mathbf{Q}_{\text{COV}} \end{aligned}$$

$$(2.18a)$$
$$\mathbf{M} = \sum_{t=1}^{n} \begin{bmatrix} \overline{\mathbf{U}}_{t}^{T} \mathbf{q}^{t^{T}} (\mathbf{m}^{1} + \dots + \mathbf{m}^{e}) \mathbf{L}_{2} \overline{\mathbf{U}}_{t} \\ \vdots \\ \overline{\mathbf{U}}_{t}^{T} \mathbf{q}^{e^{T}} (\mathbf{m}^{1} + \dots + \mathbf{m}^{e}) \mathbf{L}_{2} \overline{\mathbf{U}}_{t} \end{bmatrix}$$

$$= \sum_{t=1}^{n} \begin{bmatrix} \overline{\mathbf{U}}_{e}^{T} \mathbf{q}^{t^{T}} (\mathbf{m}^{1} + \dots + \mathbf{m}^{e}) \mathbf{L}_{2} \overline{\mathbf{U}}_{e} \\ \vdots \\ \overline{\mathbf{U}}_{e}^{T} \mathbf{q}^{e^{T}} (\mathbf{m}^{1} + \dots + \mathbf{m}^{e}) \mathbf{L}_{2} \overline{\mathbf{U}}_{e} \end{bmatrix}$$

$$+ \sum_{t=1}^{n} \begin{bmatrix} \overline{\mathbf{U}}_{n}^{T} \mathbf{q}^{t^{T}} (\mathbf{m}^{1} + \dots + \mathbf{m}^{e}) \mathbf{L}_{2} \overline{\mathbf{U}}_{n} \\ \vdots \\ \overline{\mathbf{U}}_{n}^{T} \mathbf{q}^{e^{T}} (\mathbf{m}^{1} + \dots + \mathbf{m}^{e}) \mathbf{L}_{2} \overline{\mathbf{U}}_{n} \end{bmatrix}$$

$$+ \sum_{t=1}^{n} \begin{bmatrix} \overline{\mathbf{U}}_{n}^{T} \mathbf{q}^{t^{T}} (\mathbf{m}^{1} + \dots + \mathbf{m}^{e}) \mathbf{L}_{2} \overline{\mathbf{U}}_{n} \\ \vdots \\ \overline{\mathbf{U}}_{n}^{T} \mathbf{q}^{e^{T}} (\mathbf{m}^{1} + \dots + \mathbf{m}^{e}) \mathbf{L}_{2} \overline{\mathbf{U}}_{n} \end{bmatrix}$$

$$= \mathbf{M}_{\text{exact}} + \mathbf{M}_{\text{bias}} + \mathbf{M}_{\text{COV}}$$

$$(2.18b)$$

Where  $\mathbf{Q}_{\text{exact}}$ ,  $\mathbf{Q}_{\text{bias}}$ ,  $\mathbf{Q}_{\text{COV}}$ ,  $\mathbf{M}_{\text{exact}}$ ,  $\mathbf{M}_{\text{bias}}$  and  $\mathbf{M}_{\text{COV}}$  are the exact term of  $\mathbf{Q}$ , the bias term of  $\mathbf{Q}$ , the covariance term of  $\mathbf{Q}$ , the exact term of  $\mathbf{M}$ , the bias term of  $\mathbf{M}$  and the covariance term of  $\mathbf{M}$ , respectively. Eq. (2.13a) is decomposed into the next form by Eq. (2.18a and b).

$$\mathbf{M}_{\text{exact}} + \mathbf{M}_{\text{bias}} + \mathbf{M}_{\text{COV}} + (\mathbf{Q}_{\text{exact}} + \mathbf{Q}_{\text{bias}} + \mathbf{Q}_{\text{COV}})\mathbf{E} = 0$$
(2.19)

 $Q_{exact}$  and  $M_{exact}$  are exact terms which are determined by the noise-free displacements. These terms include noise-free information and give exact solutions.

$$\mathbf{E}_{\text{exact}} = \mathbf{Q}_{\text{exact}}^{-1} \mathbf{M}_{\text{exact}}$$
(2.20)

Here,  $\mathbf{E}_{exact}$  is the vector of the exact solution for the unknown Young's modulus vector  $\mathbf{E}$ .

 $Q_{\text{bias}}$  and  $M_{\text{bias}}$  in Eq. (2.19) are bias terms which are created by the squared noise. Bias terms have the characteristic of variance because the summations of the squared noise are directly proportional to variance of the noise. These terms create biases in solutions because the noise is squared before the summation in each case. The terms do not disappear, even when the measurement times are infinite, and they converge to specific matrixes. The matrixes are proportional to the variance of the noise. Error seeds do not effect to the bias of the solutions. If the bias terms are known, the bias of the solution can be eliminated by the next equation.

$$0 = \mathbf{M} - \mathbf{M}_{\text{bias}} + (\mathbf{Q} - \mathbf{Q}_{\text{bias}})\mathbf{E}$$
(2.21)

The terms in Eq. (2.21) are defined in Eqs. (2.14a,b) and (2.18a,b).

To demonstrate the bias of the solution, a human skin tissue example is employed (Fig. 2.5). The example is modeled by  $100 \times 100$  Q4 plane strain elements and Helmholtz equation for analyzing both forward and backward problems. Exact shear modulus of normal and inclusion elements are 36kPaand 72kPa, respectively. Shear modulus of each elements are estimated by



Eq. (2.13b). Figs.  $(2.6) \sim (2.8)$  show the results on the line (A-A') of Fig. (2.5). Fig. (2.6) shows the biases from different error rates. A greater error rate results in greater bias. Fig. (2.7) shows that the bias is not affected by the noise sets. Even when the noise seeds are different, the same variance results in the same bias. Fig. (2.8) shows the solutions using Eq. (2.21). The biases are removed, but illposedness remains in the solutions. This ill-posedness occurs from the covariance terms.





Figure 2.7 Biases from different error seeds (same variance)



Figure 2.8 Ill-posedness of the solution (bias-compensation is applied)

 $Q_{COV}$  and  $M_{COV}$  are the covariance terms. These terms come from the interaction between the noise and the noise-free displacement, meaning that the covariance terms are proportional to the covariances between the noise and the noise-free displacement. These terms disappear when the measurement time is infinite, as the covariance between the white noise and the other signals is zero in each case. However, the terms are never removed with a finite measurement time. Under such a condition, the terms are completely unknown and are not predictable. No schemes have been devised to reconstruct the covariance terms. The ill-posedness of inverse analysis is caused by the randomness of the covariance terms. SI schemes which are defined by a minimization problem contain ill-posedness in

their solutions [Bui 1994, Hansen 1998]. Fig. (2.8), which is a solution using the bias compensation, shows the randomness of solutions containing ill-posedness. The instability of the solution can be explained by the lack of information. Noise in the measurement and modeling errors always exist with inverse analysis, and they cause information losses. The non-uniqueness and discontinuity of the solution are caused by a lack of information and cause instability in the solution. Regularization schemes are existing for stabilizing the solutions. The regularization schemes impose prior information to handle the lack of information. This issue is minutely discussed in section 2.5.

### 2.4 Bias Compensation

The inverse analysis using EEE is disturbed not only by ill-posedness but also by biases [Hjelmstad 1995]. The biases of the solutions are caused by error terms in Eq. (2.19). The biases are not eliminated even when the measurement time is long enough. Biases in solutions exist in all types of inverse analysis using EEE, not only in the inverse analysis for linear elastic continuum using dynamic displacement. However, the biases of the solutions are not seriously considered in many studies. The adaptive filters employ the inverse analysis using EEE for realtime processes [Haykin 2008]. Although the reference does not use 'EEE' as the name of the error estimator, the process for estimating error of the target system is identical to EEE. The result of the process has bias, but it is very small and negligible when the noise is properly taken into account by noise suppression filters. Additionally, it is easy to remove the bias from the solutions, since only the noise variances have relationship with the bias in each case. The extraction of the flutter derivatives uses the inverse analysis using EEE [Hong 2012, Cha 2015]. This method includes the bias of the solutions as well, but it is ignorable for the same reason given with the adaptive filter [Cha 2015].

However, the bias of EEE for the inverse analyses of linear elastic continuum is not ignorable because the noise is amplified twice in the spatial and temporal differentiation processes. The bias of the solution is eliminated by Eq. (2.21). If the noise values are known, the bias terms are also known by Eq. (2.12a and b). However, the noise values are always unknown and the bias terms are unknown as well.

Bias compensation is suggested to handle the bias with reconstructed bias terms. The bias compensation reconstructs the bias terms by the variances instead of with the noise values themselves. The bias compensation requires three assumptions. The first is that the noise is white noise. The second is that correlation between colored noise from different white noise is zero. The final is that the measured data is long enough and the bias terms are converged enough. The bias terms in Eqs. (2.18a and b) consist of the colored noise vectors  $\mathbf{u}_n^i$  or  $\mathbf{L}_2\mathbf{u}_n^i$ , and the square matrixes  $\mathbf{q}^{i^T}\mathbf{q}^e$  or  $\mathbf{q}^{i^T}\mathbf{m}^e$ . The colored noise vectors are represented by  $\mathbf{c}$ , and the square matrix is represented by  $\mathbf{A}$ .

$$\sum_{l=1}^{t_n} \left( \mathbf{c}_k^T \mathbf{A} \mathbf{c}_l \right) = sum \left[ \sum_{l=1}^{t_n} \mathbf{A} \circ \left( \mathbf{c}_k \mathbf{c}_l^T \right) \right]$$
(2.22a)

$$= sum \left[ \mathbf{A} \circ \sum_{t=1}^{t_n} \left( \mathbf{c}_k \mathbf{c}_l^T \right) \right]$$
(2.22b)

$$= sum \left[ \mathbf{A} \circ t_{n} \begin{pmatrix} \sigma_{k,l}^{2}(1,1) & \sigma_{k,l}^{2}(1,2) & \dots & \sigma_{k,l}^{2}(1,m) \\ \sigma_{k,l}^{2}(2,1) & \sigma_{k,l}^{2}(2,2) & \dots & \sigma_{k,l}^{2}(2,m) \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{k,l}^{2}(n,1) & \sigma_{k,l}^{2}(n,2) & \dots & \sigma_{k,l}^{2}(n,m) \end{pmatrix} \right]$$
(2.22c)

Here,  $\circ$  and  $sum[\cdot]$  denote the element multiplication of the matrixes and the summation of all elements in a matrix, respectively.  $\sigma_{k,l}^2(n,m)$  is the covariance between the n<sup>th</sup> element of the colored noise vector  $\mathbf{c}_k$  and the m<sup>th</sup> element of the colored noise vector  $\mathbf{c}_k$  and the m<sup>th</sup> element of the colored noise vector  $\mathbf{c}_l$ .  $t_n$  is the number of time steps.

When the noise is white noise,  $\mathbf{Q}_{\text{bias}}$  and  $\mathbf{M}_{\text{bias}}$  in Eqs. (2.18a and b) are reconstructed as  $\overline{\mathbf{Q}}_{\text{bias}}$  and  $\overline{\mathbf{M}}_{\text{bias}}$  by Eq. (2.22c).

$$\overline{\mathbf{Q}}_{\text{bias}} = t_n \begin{bmatrix} \sum_{m} \sigma_m^2 q_{m,m}^{1,1} & \sum_{m} \sigma_m^2 q_{m,m}^{1,2} & \cdots & \sum_{m} \sigma_m^2 q_{m,m}^{1,e} \\ \sum_{m} \sigma_m^2 q_{m,m}^{2,1} & \sum_{m} \sigma_m^2 q_{m,m}^{2,2} & \cdots & \sum_{m} \sigma_m^2 q_{m,m}^{2,e} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{m} \sigma_m^2 q_{m,m}^{e,1} & \sum_{m} \sigma_m^2 q_{m,m}^{e,2} & \cdots & \sum_{m} \sigma_m^2 q_{m,m}^{e,e} \end{bmatrix}$$
(2.23a)

$$\overline{\mathbf{M}}_{\text{bias}} = \frac{-2}{\Delta t^2} t_n \begin{bmatrix} \sum_{m}^{m} \sigma_m^2 m_{m,m}^{1,1} + \sum_{m}^{m} \sigma_m^2 m_{m,m}^{1,2} + \dots + \sum_{m}^{m} \sigma_m^2 m_{m,m}^{1,e} \\ \sum_{m}^{m} \sigma_m^2 m_{m,m}^{2,1} + \sum_{m}^{m} \sigma_m^2 m_{m,m}^{2,2} + \dots + \sum_{m}^{m} \sigma_m^2 m_{m,m}^{2,e} \\ \vdots \\ \sum_{m}^{m} \sigma_m^2 m_{m,m}^{e,1} + \sum_{m}^{m} \sigma_m^2 m_{m,m}^{e,2} + \dots + \sum_{m}^{m} \sigma_m^2 m_{m,m}^{e,e} \end{bmatrix}$$
(2.23b)

In this equations,  $\sigma_m^2$ ,  $q_{m,n}^{i,j}$  and  $m_{m,n}^{i,j}$  are the variance values of white noise in the m<sup>th</sup> measuring point, the (m,n) element of the matrix  $\mathbf{q}^{i^T}\mathbf{q}^j$  and the (m,n)element of the matrix  $\mathbf{q}^{i^T}\mathbf{m}^j$ , respectively.  $-2\Delta t^{-2}$  of Eq. (2.23b) represents the middle coefficient of the central FDM for second order derivatives using second order accuracy.  $\overline{\mathbf{Q}}_{\text{bias}}$  and  $\overline{\mathbf{M}}_{\text{bias}}$  are substituted into  $\mathbf{Q}_{\text{bias}}$  and  $\mathbf{M}_{\text{bias}}$  of Eq. (2.21), and then the biases in the solutions are eliminated.

Figs. (2.6) and (2.8) are solutions of the EEE without/with bias compensation, respectively. In Fig. (2.8), the biases of the solutions are eliminated by the bias compensation, while ill-posedness is remained.

#### Bias Compensation without Noise Variance Information

Bias compensation requires the noise variances, but it is difficult to determine the exact values in the most cases. The bias terms are directly proportional to the noise variance and thus, the noise variances can be rebuilt from a minimization problem. The object function is the difference between the measured and the analyzed displacement.

$$\min \Pi(\boldsymbol{\sigma}^2) = \left\| \overline{u}_t - u_t(E_b(\boldsymbol{\sigma}^2)) \right\|_2^2$$
(2.24)

Where  $\sigma^2$  and  $\overline{u}_t$  are the unknown noise variance vector and the measured displacement, respectively.  $u_t(E_b(\sigma^2))$  is the analyzed displacement which is calculated by the bias compensated elastic modulus using  $\sigma^2$ .  $\|\cdot\|_2^2$  indicates the L<sub>2</sub>-norm minimization. Since the number of noise variance is identical to the number of nodes, Eq. (2.24) has too much unknowns. Thus, additional information is required for reducing the number of the unknown. A proportional noise assumption or an absolute noise assumption can be the additional information.

## 2.5 Regularization

To remove the ill-posedness of the inverse analysis, missed information must be complemented. Regularization schemes have been used for stabilizing the illposedness of the inverse analyses. Unstable information is replaced by prior information, which is imposed by regularity conditions. Material property conditions are the most commonly used information in many studies. Regularizations impose the condition in which the material property functions of the continuum must be in the  $L_2$ -function space.

Three types of regularization are typically used in many fields. The first is known as truncated singular value decomposition (TSVD) [Vogel 1986]. TSVD is not a regularization scheme, but it has the basic concept of regularization.  $\mathbf{Q}$  in

Eq. (2.13b) is divided by means of singular value decomposition (SVD).

$$\mathbf{E} = -\mathbf{Q}^{-1}\mathbf{M} = -\mathbf{V}\mathbf{\Lambda}^{-1}\mathbf{V}^{T}\mathbf{M} = -\sum_{j=1}^{n}\frac{1}{\lambda_{j}}\mathbf{v}_{j}\mathbf{v}_{j}^{T}\mathbf{M} = -\sum_{j=1}^{n}\frac{1}{\lambda_{j}}\left(\mathbf{v}_{j}^{T}\mathbf{M}\right)\mathbf{v}_{j}$$
(2.25)

Here,  $\mathbf{V}$ ,  $\mathbf{v}_j$ ,  $\mathbf{v}_j^T$ ,  $\mathbf{\Lambda}$ ,  $\lambda_j$  and n are the singular vector matrix, the  $j^{th}$  left singular vector, the  $j^{th}$  right singular vector, the singular value matrix (diagonal matrix), the  $j^{th}$  singular value and the number of unknown system parameters, respectively. The singular values and vectors are ordered in a descending order of singular values. The left singular vector matrix is a transpose of the right singular vector matrix because  $\mathbf{Q}$  is a square matrix. Small singular values and their vectors are minor factors in the input-output relationship, but they have major effects on the noise amplification process [Hansen 1998]. Zero singular values take on infinite values due to the inverse process of  $\mathbf{Q}$ . Truncating small singular values and vectors stabilizes the inverse analysis.

$$\mathbf{E} = -\mathbf{Q}^{-1}\mathbf{M} = -\sum_{j=1}^{n} \frac{1}{\lambda_{j}} \left( \mathbf{v}_{j}^{T} \mathbf{M} \right) \mathbf{v}_{j} \approx -\sum_{j=1}^{t} \frac{\mathbf{v}_{j}^{T} \mathbf{M}}{\lambda_{j}} \mathbf{v}_{j}$$
(2.26)

In this equation, t is the truncation number. The truncation number denotes the number of singular values which have useful information. Too large truncation number makes TSVD meaningless, whereas too small truncation number leads to a loss of useful information. A bilinear fitting method (BFM) is utilized to decide

the truncation number [Park 2007]. Truncated singulars contain part of the information, even if it is polluted. The truncated information is another source of noise in the solution. This noise is related to the uniqueness of the solution. Solutions from SI using TSVD may be not precise solutions but may instead be another set of solutions from the same input-output values. This indicates that even if the solution itself is incorrect, the input-output relationships from the solution is exact. Thus, TSVD is usually used with adaptive filters, signal processes and control fields. These fields focus on the input-output relationships of target systems, not systems overall.

 $L_1$ -norm regularization has been proposed to correct the non-uniqueness of TSVD [Park 2007]. Prior information replaces the truncated singular parts of TSVD.

$$\mathbf{E} = -\mathbf{Q}^{-1}\mathbf{M} \approx -\sum_{j=1}^{t} \frac{\mathbf{v}_{j}^{T}\mathbf{M}}{\lambda_{i}} \mathbf{v}_{j} + \sum_{j=t+1}^{n} \gamma_{j} \mathbf{v}_{j} = \mathbf{E}_{t} + \mathbf{E}_{r}$$
Subject to  $\mathbf{V}_{t}^{T}\mathbf{E}_{r} = 0$  and  $\mathbf{E}_{\text{lower}} - \mathbf{E}_{t} \leq \mathbf{E}_{r} \leq \mathbf{E}_{\text{upper}} - \mathbf{E}_{t}$ 

$$(2.27)$$

Here,  $\mathbf{v}_j$  and  $\gamma_j$ , which satisfy the condition of j > t, are the rebuilt singular vectors and the undetermined constants of the rebuilt vectors, respectively.  $\mathbf{E}_t$ ,  $\mathbf{E}_r$ ,  $\mathbf{E}_{lower}$  and  $\mathbf{E}_{upper}$  denote the solution by TSVD, the linear combination of the truncated singular vectors, the lower bound of the solution and the upper bound of the solution, respectively.  $\mathbf{E}_r$  is determined by prior information which is that

the stiffness functions of the continua in the spatial domain must be in the  $L_2$ -norm function space [Park 2007]. The prior condition is imposed by the  $L_1$ -norm form of the  $L_2$ -norm function space condition.

$$\min_{\mathbf{E}_r} \Psi_1 = \left\| \nabla (\mathbf{E} - \mathbf{E}_{\text{prior}}) \right\| = \left\| \nabla [\mathbf{E}_r - (\mathbf{E}_{\text{prior}} - \mathbf{E}_t)] \right\|$$
Subjected to  $\mathbf{V}_t^T \mathbf{E}_r = 0$  and  $\mathbf{E}_{\text{lower}} - \mathbf{E}_t \le \mathbf{E}_r \le \mathbf{E}_{\text{upper}} - \mathbf{E}_t$ 

$$(2.28)$$

Since the optimization problem defined in Eq. (2.28) is a linear programming with respect to  $\mathbf{E}_r$ ,  $\mathbf{L}_1$ -norm regularization is an iterative scheme which requires a considerable amount of computational time.

 $L_2$ -norm regularization is suggested to build a quadratic problem. The prior and measured information are mixed at the rate of the regularization factor and the singular values. The prior information used with  $L_1$ -norm regularization is also used in this case, but the prior information is imposed in a  $L_2$ -norm form. The prior information is imposed by means of Tikhonov regularization [Lee 1999, Park 2001].

$$\Pi_{\text{EEE}} = \sum_{k=1}^{m} \left\| \sum_{e} \mathbf{m}_{k}^{e} + \sum_{e} E^{e} \mathbf{q}_{k}^{e} \right\|_{2}^{2} + \beta \left\| \mathbf{E} - \mathbf{E}_{\text{prior}} \right\|_{2}^{2}$$
(2.29)

Eq. (2.29) is a quadratic problem when the equilibrium equation is linear with respect to  $\mathbf{E}$ . It is not an iterative problem, and it requires low computational efficiency.

The three regularization schemes above have been used in many fields. Most cases are soft inclusion cases, for which the above regularization schemes work well. However, regularization does not work properly in cases involving hard inclusion. Tumors in human bodies, suspensions of vehicles and other such examples are hard inclusion cases. Figs. (2.9) and (2.10) show the results of soft and hard inclusion cases when using  $L_2$ -norm regularization, respectively. These examples are the ultrasonic elastography examples discussed in section 2.3.2 (Fig. 2.5). In the soft inclusion case,  $L_2$ -norm regularization gives a useful solution. On the other hand, the hard inclusion information is truncated by  $L_2$ -norm regularization.

To understand the drawbacks of regularization, the mechanism of regularization must be understood. Noise-free information of  $\mathbf{Q}$  in Eq. (2.13b) is decomposed into two parts, i.e., singular values and singular vectors, by SVD. The singular values contain the scale information of the elements in  $\mathbf{Q}$ , and the singular vectors, which are unit vectors, have the position information of the scale information. Smaller singular values are related to larger values of  $\mathbf{E}$  (harder elements of the FEM), and larger singular values are related to smaller values of  $\mathbf{E}$  (softer elements of the FEM), because  $\mathbf{Q}$  is going to be the inverse matrix  $\mathbf{Q}^{-1}$  with which to calculate  $\mathbf{E}$ . When these properties of  $\mathbf{Q}$  and  $\mathbf{E}$  factor into the process of regularization, the hard inclusion information is truncated. Small singular values and their vectors are removed/replaced/weakened



Figure 2.9 L2-norm regularization result of the soft inclusion case



Figure 2.10 L<sub>2</sub>-norm regularization result of the hard inclusion case



Figure 2.11 Mechanism of regularizations

(TSVD/ $L_1/L_2$ , respectively) by regularization and amplified noise is removed/replaced/weakened. During this process, information about hard elements, which is included in the small singular value parts, is also removed/replaced/weakened. It indicate that regularization is not valid for cases involving hard inclusion. The procedure above is concluded in Fig. (2.11).

The regularizations are valid in typical cases because most damage cases involve soft inclusion. In some cases, however, involving hard inclusion, the regularizations are not valid. In the next section, a new stabilization schemes are proposed to prevent the side effects associated with regularization.

# 3. Acceleration-Energy Filter

Noise filters using regularity conditions uses physical laws on the measurements as a prior condition. The physical laws of equilibrium equation (Eq. 2.4) must be satisfied, but noise in the measured displacement breaks the physical laws. This situation is caused by differentiation sequences in the inverse analysis of the continua, which amplify high-frequency noise. Amplified high-frequency noise acts like a series of Dirac delta functions and breaks the physical laws. The measured displacements without the physical laws effect the ill-posedness of inverse analyses. The physical laws of Eq. (2.4) must be guaranteed to avoid illposedness.

## 3.1 Noise Filter Using Regularity Conditions of Displacement

To prevent ill-posedness of the inverse analysis, the measured displacements must satisfy the physical laws of the equilibrium equation. Eq. (2.4) in terms of measured displacement  $\overline{u}$  is given below.

$$C_{ijkl} \frac{\partial^2 \overline{u}_k}{\partial x_i \partial x_l} - \rho \frac{\partial^2 \overline{u}_i}{\partial t^2} = 0$$
(3.1)

The measured displacement  $\overline{u}$  have to satisfy two physical laws, the finiteness of the strain energy and the acceleration, to satisfy the equilibrium

equation (Eq. 3.1).

The next equation represents the first law, the finiteness of the strain energy.

$$\int_{V} \frac{d\overline{u}_{i}}{dx_{i}} C_{ijkl} \frac{d\overline{u}_{k}}{dx_{l}} dV < \infty$$
(3.2)

The equation describes that total internal energy of continuum must be finite. The next equation represents the second law, the finiteness of the acceleration.

$$\int_{T} \left\| \frac{d^{2} \overline{u}_{i}}{dt^{2}} \right\|_{2}^{2} dt < \infty$$
(3.3)

The equation describes that the second derivative of the measured displacement must be finite. In fact, acceleration functions are not finite functions, because the Dirac delta functions can be included in the acceleration functions. Except for the Dirac delta functions, however, acceleration functions exist in the finite function space. While the impact loads are negligible, it is a valid assumption that acceleration functions exist in the finite function space.

The measured displacement with noise, however, cannot satisfy above two physical laws. Because the amplified high-frequency noise sources, which are amplified by differentiations, act as a series of Dirac delta functions (section 2.3.1). A series of Dirac delta functions breaks the physical laws, and causes ill-posedness of the inverse analysis.

Noise filtering using regularity conditions can prevent the dissatisfaction of

the physical laws. Filtered displacement  $\tilde{u}$  is defined by three conditions. The first condition is that the filtered displacement must sticks around the measured displacement.

$$\min \Pi = \frac{1}{2} \int_{T} \int_{V} (\widetilde{u}_{i} - \overline{u}_{i})^{2} dV dt , \qquad (3.4)$$

where  $\tilde{u}$  and  $\bar{u}$  denote the filtered displacement and the measured displacement, respectively. The second and the third conditions are regularity conditions using the physical laws of the displacement.

$$\Pi_{e} = \int_{t} \int_{V} \frac{d\widetilde{u}_{i}}{dx_{j}} C_{ijkl} \frac{d\widetilde{u}_{k}}{dx_{l}} dV dt < \infty, \qquad (3.5)$$

$$\Pi_{a} = \int_{T} \int_{V} \left\| \frac{d^{2} \widetilde{u}_{i}}{dt^{2}} \right\|_{2}^{2} dV dt < \infty$$
(3.6)

Eqs. (3.5) and (3.6) represent the regularity conditions using the finiteness of the strain energy and the accelerations, respectively. The regularity conditions are enforced to Eq. (3.4) as penalty functions.

$$\min_{u_i} \Pi = \frac{1}{2} \int_T \int_V (\widetilde{u}_i - \overline{u}_i)^2 dV dt 
+ \frac{\lambda_a}{2} \int_T \int_V \left\| \frac{d^2 \widetilde{u}_i}{dt^2} \right\|_2^2 dV dt + \frac{\lambda_e}{2} \int_T \int_V \frac{d\widetilde{u}_i}{dx_j} C_{ijkl} \frac{d\widetilde{u}_k}{dx_l} dV dt$$
(3.7)

Here,  $\lambda_a$  and  $\lambda_e$  are regularization factors which define the ratio between the

regularity and measured information. Since The object function is designed to suppress noise which break the physical laws, the minimization problem is considered as a noise filter. A governing equation and boundary conditions help to figure out characteristics of the acceleration-energy filter. The variational principle and the integration by parts are applied to Eq. (3.7) for deriving the governing equation and the boundary conditions.

$$\left. \delta \widetilde{u}_i C_{ijkl} \frac{d\widetilde{u}_k}{dx_l} n_j \right|_{S} = 0 \tag{3.8d}$$

$$\frac{d^3 \tilde{u}_i}{dt^3} = \frac{da_i}{dt}, \quad \frac{d^2 \tilde{u}_i}{dt^2} = a_i \quad \text{at} \quad t = t_i, t_f$$
(3.8e)

Where  $\delta$ ,  $n_j$ ,  $t_i$  and  $t_f$  are a variation, a nominal vector, an initial measuring time and a final measuring time, respectively. Eqs. (3.8a and b) are the sequence of the variational principle and the integration by parts, respectively. Eq. (3.8b)

gives the governing equation (Eq. 3.8c), the boundary condition of the spatial domain (Eq. 3.8d) and the initial/final value conditions of the temporal domain (Eq. 3.8e). The spatial boundary condition is eliminated at the free and fixed boundaries, because it is identical to the boundary condition of the continua. The temporal boundary condition is not eliminated and must be considered.

## 3.2 Characteristics of Acceleration-Energy Filter

#### 3.2.1 Characteristics in Frequency Domain

A transfer function is useful for identifying the characteristics of the filtering process [Hamming 1989]. The transfer function, which shows the input-output relationship in the frequency domain, is defined as the output divided by the input in the frequency domain (Eq. 2.16). The transfer function for the accelerationenergy filter is derived from the Fourier transformed governing equation of the filter (Eq. 3.8c). However, the filter is a four-dimensional filter and hard to be analyzed. To simplifying Eq. (3.8c) the equilibrium equation for the continuum (Eq. 2.1) is substituted into the last term of Eq. (3.8c). The body force  $b_i$  is banished by the first assumption of Section 2.1. The equation below is a simplified governing equation.

$$\widetilde{u}_{i} - \overline{u}_{i} + \lambda_{a} \frac{d^{4}\widetilde{u}}{dt^{4}} - \lambda_{e} C_{ijkl} \frac{d^{2}\widetilde{u}_{k}}{dx_{j} dx_{l}} = u_{i} - \overline{u}_{i} + \lambda_{a} \frac{d^{4}\widetilde{u}}{dt^{4}} - \lambda_{e} \rho \frac{d^{2}\widetilde{u}}{dt^{2}} = 0$$
(3.9)

The Fourier transform and the transfer function of the above equation is shown below.

$$\mathcal{F}(\overline{u}) = (1 + \lambda_a \omega_t^4 + \lambda_e \rho \omega_t^2) \mathcal{F}(\widetilde{u})$$
(3.10)

$$H(\omega_t) = \frac{\mathcal{F}(\tilde{u})}{\mathcal{F}(\bar{u})} = \frac{1}{1 + \lambda_a \omega_t^4 + \lambda_e \rho \omega_t^2} = \frac{1}{1 + 16\lambda_a \pi^4 f^4 + 4\lambda_e \pi^2 \rho f^2}$$
(3.11)

Here,  $\mathcal{F}(\vec{u})$ ,  $\mathcal{F}(\vec{u})$ ,  $H(\omega_t)$ ,  $\omega_t$  and f are the Fourier transform of the measured displacement, the Fourier transform of the filtered displacement, the transfer function, the temporal frequency in radians and the temporal frequency in Hz, respectively. It is a monotonically decreasing function from H(0) = 1 to  $H(\infty) = 0$ . The transfer function of the acceleration- energy filter consists of the pass band, the cut-off band and the transient which are a signal-conserving range, a signal-eliminating range and a transient between the above two ranges, respectively. The transfer function in the pass band has value of near one, while the transfer function in the cut-off band has value of near zero. The acceleration-energy filter in this case is a low-pass filter, which is typically used to stabilize numerical differentiations. This is feasible because amplified high-frequency noise is the greatest problem, and a low-pass transfer function suppresses high-frequency noise. However, this instance of the filter has strong physical meaning and is thus optimized for the inverse analysis of the continuum.

A normalized transfer function with respect to a target frequency  $f_t$  is

employed for simplifying characteristics of the acceleration-energy filter. The target frequency  $f_t$  represents an upper bound of the pass band. The target frequency, the upper bound of the pass band, is chosen as the largest valid frequency of the measured displacement.

$$H(f) = \tilde{H}(\tilde{f}) = \frac{1}{1 + 16\lambda_a \pi^4 \tilde{f}^4 f_t^4 + 4\lambda_e \pi^2 \rho \tilde{f}^2 f_t^2} = \frac{1}{1 + \lambda_{a,2} \tilde{f}^4 + \lambda_{e,2} \tilde{f}^2}$$
(3.12)

Here,  $\tilde{H}(\tilde{f})$ ,  $f_t$ ,  $\tilde{f} = f/f_t$ ,  $\lambda_{e,2} = 4\lambda_e \pi^2 \rho f_t^2$  and  $\lambda_{a,2} = 16\lambda_t \pi^4 f_t^4 \lambda_a$  are the normalized transfer function, the target frequency, the dimensionless normalized frequency, the strain energy regularization factor of normalized transfer function and the acceleration regularization factor of normalized transfer function, respectively. The characteristics of the acceleration-energy filter are defined by the regularization factors  $\lambda_{e,2}$  and  $\lambda_{a,2}$ .

#### 3.2.2 Separation of Regularity Conditions

The given filter, Eq. (3.7), has both temporal and spatial calculation, but it is difficult to handle both dimensions at once. In this section, Eq. (3.7) will be approximately separated into temporal and spatial parts in the frequency domain. The transfer function of the filter (Eq. 3.12) is useful for simplifying the minimization. The next equation is an approximation of Eq. (3.7).

$$\frac{\mathcal{F}(\tilde{u})}{\mathcal{F}(\bar{u})} = H(f) = \frac{1}{1 + \lambda_{a,2}\tilde{f}^4 + \lambda_{e,2}\tilde{f}^2}$$

$$\approx \frac{1}{1 + \lambda_{a,2}\tilde{f}^4 + \lambda_{e,2}\tilde{f}^2 + \lambda_{a,2}\lambda_{e,2}\tilde{f}^6}$$

$$= \frac{1}{1 + \lambda_{a,2}\tilde{f}^4} \frac{1}{1 + \lambda_{e,2}\tilde{f}^2}$$
(3.13)

The term  $\lambda_{a,2}\lambda_{e,2}\tilde{f}^6$  is added to the denominator of the transfer function. The transfer function shows that the filter is a low-pass filter, which consists of a pass band, a cut-off band and a transient. In the pass band, the added term is nearly zero. In the transient and the cut-off band, the added term accelerates the convergence of the transfer function. It means that the approximated transfer function can replace the original one. Because the regularization factors  $\lambda_{e,2}$  and  $\lambda_{a,2}$  are not defined yet, accuracy of approximated transfer function will be analytically discussed in Section 3.2.4.

The series of ensuing equations separates the approximated equation into two regularizations.

$$(1 + \lambda_{a,2}\tilde{f}^{4})(1 + \lambda_{e,2}\tilde{f}^{2})\mathcal{F}(\tilde{u}) = \mathcal{F}(\bar{u})$$
(3.14a)

$$(1+\lambda_{a,2}\tilde{f}^{4})\mathcal{F}(\tilde{u}) = \mathcal{F}(\tilde{u}) \quad \& \quad (1+\lambda_{e,2}\tilde{f}^{2})\mathcal{F}(\tilde{u}) = \mathcal{F}(\bar{u}) \tag{3.14b,c}$$

$$(1 + \lambda_a \omega_t^4) \mathscr{F}(\widetilde{u}) = \mathscr{F}(\widetilde{u}) \quad \& \quad (1 + \lambda_e \rho \omega_t^2) \mathscr{F}(\widetilde{u}) = \mathscr{F}(\widetilde{u}) \tag{3.14d,e}$$

$$0 = \tilde{u}_i - \breve{u}_i + \lambda_a \frac{d^4 \tilde{u}}{dt^4} \quad \& \quad 0 = \breve{u}_i - \overline{u}_i - \lambda_e \rho \frac{d^2 \breve{u}}{dt^2}$$
(3.14f,g)

$$0 = \widetilde{u}_i - \widetilde{u}_i + \lambda_a \frac{d^4 \widetilde{u}}{dt^4} \quad \& \quad 0 = \widetilde{u}_i - \overline{u}_i - \lambda_e C_{ijkl} \frac{d^2 \widetilde{u}_k}{dx_j dx_l}$$
(3.14h,i)

Where  $\overline{u}$ ,  $\overline{u}$  and  $\widetilde{u}$  are the measured displacements, the filtered displacements by the strain energy regularity condition and the filtered displacement by both the strain energy/acceleration condition. The commutative law of both conditions is valid since multiplication of the two transfer function in Eq. (3.14a) satisfies the commutative law. The inverse processes of the integration by parts of Eqs. (3.14h and i) give the object function of the equations.

$$\min \Pi(u) = \frac{1}{2} \int_{T} (\widetilde{u}_{i} - \widetilde{u}_{i})^{2} dt + \frac{\lambda_{a}}{2} \int_{T} \left\| \frac{d^{2} \widetilde{u}_{i}}{dt^{2}} \right\|_{2}^{2} dt$$
(3.15a)

$$\min \Pi = \frac{1}{2} \int_{V} (\breve{u}_{i} - \overline{u}_{i})^{2} dV + \frac{\lambda_{e}}{2} \int_{V} \frac{d\breve{u}_{i}}{dx_{j}} C_{ijkl} \frac{d\breve{u}_{k}}{dx_{l}} dV$$
(3.15b)

The noise filter is separated into the temporal and the spatial filters, or the acceleration and the energy filters. The integral dimensions are also separated into their temporal and spatial dimensions. Doing this is easier than the original four-dimension integration.

#### 3.2.3 Characteristics and Regularization Factors of Acceleration Filter

Since Eqs. (3.15a and b) satisfy the commutative law, the acceleration filter is independent from the energy filter. The acceleration filter has own input and output displacement, and the object function (Eq. 3.15a) changes into the next

equation.

$$\min \Pi(u) = \frac{1}{2} \int_{T} (\overline{u}_{i} - \widetilde{u}_{i})^{2} dt + \frac{\lambda_{a}}{2} \int_{T} \left\| \frac{d^{2} \widetilde{u}_{i}}{dt^{2}} \right\|_{2}^{2} dt$$
(3.16)

Here,  $\tilde{u}_i$  and  $\bar{u}_i$  are the output (filtered) displacement and the input (measured or output from the energy filter) displacement, respectively. A transfer function of the acceleration filter is derived from a variation of Eq. (3.16)

$$\delta\Pi = \int_{T} \int_{V} \left[ \delta \widetilde{u}_{i} (\widetilde{u}_{i} - \overline{u}_{i}) + \lambda_{a} \frac{d^{2} \delta \widetilde{u}_{i}}{dt^{2}} \frac{d^{2} \widetilde{u}_{i}}{dt^{2}} \right] dV dt \qquad (3.17a)$$

$$= \int_{T} \int_{V} \delta \widetilde{u}_{i} \left[ \widetilde{u}_{i} - \overline{u}_{i} + \lambda_{a} \frac{d^{4} \widetilde{u}}{dt^{4}} \right] V dt$$

$$+ \lambda_{a} \left[ \int_{V} \frac{d \delta \widetilde{u}_{i}}{dt} \left( \frac{d^{2} \widetilde{u}_{i}}{dt^{2}} - a \right) dV \right]_{t_{i}}^{t_{f}} - \lambda_{a} \left[ \int_{V} \delta \widetilde{u}_{i} \left( \frac{d^{3} \widetilde{u}_{i}}{dt^{3}} - \frac{da}{dt} \right) dV \right]_{t_{i}}^{t_{f}} \qquad (3.17b)$$

Here,  $\tilde{u}_i$ ,  $\bar{u}_i$ ,  $t_i$  and  $t_f$  are the filtered displacement, the input displacement, an initial measuring time and a final measuring time, respectively. The inner equation in the first term of Eq. (3.17b) is the governing equation of the acceleration filter and the other terms are the boundary conditions.

$$0 = \tilde{u} - \bar{u} + \lambda_a \frac{d^4 \tilde{u}}{dt^4}$$
(3.18a)

$$\frac{d^{3}\widetilde{u}_{i}}{dt^{3}} = \frac{da_{i}}{dt}, \quad \frac{d^{2}\widetilde{u}_{i}}{dt^{2}} = a_{i} \quad \text{at} \quad t = t_{i}, t_{f}$$
(3.18b)

Eqs. (3.18a and b) are the governing equation and the boundary conditions, respectively. The transfer function of the acceleration filter is derived from the transfer function of Eq. (3.18a).

$$H_{a}(f) = \frac{1}{1 + 16\lambda_{a}\pi^{4}f}$$
(3.19a)

$$\widetilde{H}_{a}(\widetilde{f}) = \frac{1}{1 + \lambda_{a,2}\widetilde{f}^{4}}$$
(3.19b)

Here,  $H_a(f)$ ,  $\tilde{H}_a(\tilde{f})$ ,  $f_t$ ,  $\tilde{f} = f/f_t$  and  $\lambda_{a,2} = 16\lambda_a \pi^4 f_t^4$  are the original transfer function of the acceleration filter, the normalized transfer function of the acceleration filter, the target frequency, the dimensionless normalized frequency and the regularization factor of normalized transfer function, respectively. It is a monotonically decreasing function from  $H_a(0) = 1$  to  $H_a(\infty) = 0$ . The acceleration filter in this case is a low-pass filter, which is typically used to stabilize numerical differentiations. This is feasible by same reason of the acceleration-energy filter. However, this instance of the filter has strong physical meaning and is thus linked to the energy filter (Eq. 3.15b). The low-pass filter consists of the pass band, the cut-off band and the transient which are a signal-conserving range, a signal-eliminating range and a transient between the above two ranges, respectively.

Eq. (3.20b) is a normalized transfer function with respect to the target frequency  $f_t$ . The normalized transfer function is employed for simplifying characteristics of the acceleration filter. As mentioned in Section 3.2.1, The target

frequency  $f_t$  represents upper bound frequency of the pass band and is chosen as the largest valid frequency of the measured displacement.

The regularization factor is defined by a target accuracy  $\tilde{H}(1) = H_t$ , which is the desired accuracy for the target frequency  $\tilde{f} = 1$ .

$$\lambda_{t,2} = \frac{1}{H_t} - 1$$
 when  $0 < H_t < 1$  (3.20)

Since the target frequency cannot be 0 nor an infinite value, the target accuracy must be in  $0 < H_t < 1$ . The target accuracy must be pre-defined by an engineering sense to determined the regularization factor. Here, the values of 0.97, 0.95 and 0.90 are recommended. Fig.(3.1) shows the normalized transfer functions of the acceleration filter for various levels of the target accuracy. The higher target accuracy gives the better pass band and the worse cut-off band, and vise versa.

#### 3.2.3 Characteristics and Regularization Factors of Energy Filter

Most noise sources are suppressed by the acceleration regularization, but the noise associated with the pass band remains. The remaining noise is white noise in the spatial direction. These sources are amplified by the strain-displacement relationship and render the signals such that they are not satisfying the physical laws of continuum. The energy filter (Eq. 3.15b) is employed to guarantee the



physical laws of the strain energy. The energy filter, which is separated from the acceleration-energy filter, is independent from the acceleration filter. The energy filter has own input and output displacement, and thus, the object function (Eq. 3.15b) changes into the next equation.

$$\min \Pi = \frac{1}{2} \int_{V} (\tilde{u}_{i} - \overline{u}_{i})^{2} dV + \frac{\lambda_{e}}{2} \int_{V} \frac{d\tilde{u}_{i}}{dx_{i}} C_{ijkl} \frac{d\tilde{u}_{k}}{dx_{l}} dV$$
(3.21)

Here,  $\tilde{u}_i$  and  $\bar{u}_i$  are the output (filtered) displacement and the input (measured /

output from the acceleration filter) displacement, respectively. A transfer function of the energy filter is derived from a variation of Eq. (3.21).

$$\delta \Pi = \int_{V} \left[ \delta \widetilde{u}_{i} (\widetilde{u}_{i} - \overline{u}_{i}) + \lambda_{e} \frac{d \delta \widetilde{u}_{i}}{dx_{j}} C_{ijkl} \frac{d \widetilde{u}_{k}}{dx_{l}} \right] dV$$
(3.22a)

$$= \int_{V} \delta \widetilde{u}_{i} \left[ \widetilde{u}_{i} - \overline{u}_{i} - \lambda_{e} C_{ijkl} \frac{d^{2} \widetilde{u}_{k}}{dx_{j} dx_{l}} \right] dV + \lambda_{e} \int_{S} \delta \widetilde{u}_{i} C_{ijkl} \frac{d \widetilde{u}_{k}}{dx_{l}} n_{j} dS \qquad (3.22b)$$

The inner equation in the first term of Eq. (3.22b) is the governing equation of the energy filter and the second term is the boundary condition.

$$\widetilde{u}_i - \overline{u}_i - \lambda_e C_{ijkl} \frac{d^2 \widetilde{u}_k}{dx_i dx_l} = 0$$
(3.23a)

$$\delta \tilde{u}_i C_{ijkl} \left. \frac{d \tilde{u}_k}{d x_l} n_j \right|_{s} = 0 \tag{3.23b}$$

Eqs. (3.23a and b) are the governing equation and the boundary condition, respectively. The boundary condition is identical to the boundary condition of the continua; it is zero on fixed and traction-free boundaries. The governing equation is similar to that during acceleration regularization, i.e., functioning as a low-pass and high-cut filter. However, it is too complex to analyze the energy filter in the spatial frequency domain, since the elastic waves (the s-wave, the p-wave, etc.) have physical relationships. Moreover, the elastic waves have relationship with the temporal frequency. The next equations show relation between the spatial frequencies of the elastic waves and the temporal frequency.

$$\omega_{\text{swave}} = \omega_t \sqrt{\frac{\rho}{\mu}}$$
(3.24a)

$$\omega_{\text{pwave}} = \omega_t \sqrt{\frac{\rho}{\lambda + 2\mu}}$$
(3.24b)

Where  $\omega_{swave}$ ,  $\omega_{pwave}$ ,  $\omega_t$ ,  $\rho$ ,  $\lambda$  and  $\mu$  are the spatial frequency of the swave, the spatial frequency of the p-wave, the temporal frequency, the mass density of the medium, the Lame's first parameter of the medium and the Lame's second parameter of the medium. The Lame's second parameter  $\mu$  represents the shear modulus when the medium is a continuum.

To simplify frequency characteristics of the energy filter, the equilibrium equation for the continuum (Eq. 2.1) is substituted into the last term of Eq. (3.23a). The equilibrium equation for the continuum includes information of the relationship between the elastic waves and the temporal wave, due to the fact that Eqs. (3.24a and b) are come from the equilibrium equation. The body force  $b_i$  is banished by the first assumption of Section 2.1. It is the same sequence to section 3.2.1. The equation below is a simplified governing equation.

$$u_i - \overline{u}_i - \lambda_e \rho \frac{d^2 \widetilde{u}}{dt^2} = 0$$
(3.25)

The Fourier transform and the transfer functions of above equation are below.

$$\mathcal{F}(\overline{u}) = (1 + \lambda_e \rho \omega_t^2) \mathcal{F}(\widetilde{u})$$
(3.26)

$$H_{e}(\omega_{t}) = \frac{\mathcal{F}(\widetilde{u})}{\mathcal{F}(\overline{u})} = \frac{1}{1 + \lambda_{e}\rho\omega_{t}^{2}} = \frac{1}{1 + 4\lambda_{e}\pi^{2}\rho f^{2}}$$
(3.27)

Here,  $\mathcal{F}(\overline{u})$ ,  $\mathcal{F}(\widetilde{u})$ ,  $H_e(\omega)$ ,  $\omega_t$  and f are the Fourier transform of the measured displacement, the Fourier transform of the filtered displacement, the transfer function of the energy filter, the frequency in radians and the frequency in Hz, respectively. It is also a monotonically decreasing function from H(0) = 1 to  $H(\infty) = 0$ .

Since the elastic waves and the temporal wave are closely related by the equilibrium equation (or Eqs. 3.24a and b), valid frequency ranges of the energy filter and the acceleration filter must be identical. It means that two filters must have the consistent pass band. The solution of the inverse analysis is not precise without consistency between the acceleration filter and the energy filter, because the temporal and spatial derivatives contain different information when the consistency requirement is not satisfied. Since the pass band is defined by the target frequency and the target accuracy, the consistency between two filters is guaranteed by the same target frequency and accuracy. A normalized transfer function of the energy filter with respect to the  $f_t$  is in the next

$$\tilde{H}_{e}(\tilde{f}) = \frac{1}{1 + \lambda_{e,2}\tilde{f}^{2}}$$
(3.28)

Here,  $\lambda_{e,2} = 4\lambda_e \pi^2 \rho f_t^2$  is a normalized regularization factor of the energy filter. The normalized regularization factor is defined by a target accuracy  $\tilde{H}(1) = H_t$ , which is the desired accuracy for the target frequency  $\tilde{f} = 1$ , and is identical to the normalized regularization factor of the acceleration filter  $\lambda_{a,2}$ .

$$\lambda_{e,2} = \frac{1}{H_t} - 1$$
 when  $0 < H_t < 1$  (3.29)

A normalized regularization factor  $\lambda_2$  is employed for representing both regularization factors of the acceleration and energy filters.

$$\lambda_2 = \lambda_{a,2} = \lambda_{e,2} = \frac{1}{H_t} - 1$$
 when  $0 < H_t < 1$  (3.30)

Fig.(3.2) shows the normalized transfer functions of the energy filter for various levels of the target accuracy. The higher target accuracy gives the better pass band and the worse cut-off band, and vise versa.

#### 3.2.4 Characteristics of Acceleration-Energy Filter

The regularization factors  $\lambda_{a,2}$  and  $\lambda_{e,2}$  in the transfer function of the acceleration-energy filter (Eq. 3.13) are determined by Eq. (3.30). The original transfer function of the acceleration-energy filter in terms of the target accuracy



 $H_t$  is below.

$$\widetilde{H}(\widetilde{f}) = \frac{H_t}{H_t + (1 - H_t)\widetilde{f}^4 + (1 - H_t)\widetilde{f}^2}$$
(3.31)

The approximated transfer function in terms of the target frequency  $H_t$  is below.

$$\widetilde{H}_{approx}(\widetilde{f}) = \frac{H_t}{H_t + (1 - H_t)\widetilde{f}^4} \frac{H_t}{H_t + (1 - H_t)\widetilde{f}^2}$$
(3.32)

 $\tilde{H}_{approx}(\tilde{f})$  is the approximation of the normalized transfer function for the acceleration-energy filter. Figs. (3.3a and b) shows shapes of the original and approximated transfer functions for various target accuracies,  $H_t = 0.97$ ,  $H_t = 0.95$  and  $H_t = 0.90$ . As target accuracy increases, accuracy of pass band also increases, while noise suppression effect decrease.

Figs. (3.4, 3.5 and 3.6) compare the original and approximated transfer functions for  $H_t = 0.97$ ,  $H_t = 0.95$  and  $H_t = 0.90$ , respectively. The figures show that difference between the both transfer functions is acceptable for the recommended target accuracies.

In Figs. (3.3~3.6), the target accuracies of the original and approximated acceleration-energy filters ( $H_{t,ori}$  and  $H_{t,approx}$ , respectively) are different from the target accuracy of the acceleration filter and the energy filter  $H_t$ . Relationship between  $H_{t,ori}$  or  $H_{t,approx}$ , and  $H_t$  is derived from Eqs. (3.31 and 3.32) for  $\tilde{f} = 1$ .

$$H_{t,ori} = \frac{H_t}{2 - H_t} \tag{3.33}$$

$$H_{t,approx} = H_t^2 \tag{3.34}$$

Fig. (3.7) shows the relationship between the target accuracies of the separated filters, the original filter, and the approximated filter. Difference between the target accuracies are acceptable when  $H_t$  is near to one.






Figure 3.4 Normalized transfer function of acceleration-energy filter for  $H_t$ =9.7 (a) small scale (b) detail of target frequency



Figure 3.5 Normalized transfer function of acceleration-energy filter for  $H_t$ =9.5 (a) small scale (b) detail of target frequency



Figure 3.6 Normalized transfer function of acceleration-energy filter for  $H_t$ =9.0 (a) small scale (b) detail of target frequency



Figure 3.7 Relationship between  $H_t$ ,  $H_{t,ori}$  and  $H_{t,approx}$ 

# 3.3 Discretization of Filters

## 3.3.1 Discretization of Acceleration Filter

A Conventional finite impulse response (CFIR) filter is typically used to approximately discretize a transfer function on the frequency domain. The CFIR filter is a linear filter, which is a combination of input and filter coefficients [Hamming 1989].

$$\widetilde{u}_{t} = \widetilde{u}(t) = \sum_{p=-k}^{k} c_{p} \overline{u}_{t+p} = \sum_{p=-k}^{k} c_{p} \overline{u}(t+p\Delta t)$$
(3.35)

Where k,  $c_p$ ,  $\tilde{u}_t$  and  $\bar{u}_t$  are the truncation number, the coefficients of CFIR filter, the output (filtered) displacement on the discretized time t and the input (measured / output from the energy filter) displacement on the discretized time t, respectively. The transfer function of the CFIR filter is derived by the Fourier transform of Eq. (3.35).

$$H_t(f) \approx H_{CFIR}(f) = \sum_{p=-k}^k c_p \mathrm{e}^{\mathrm{i}2\pi f p\Delta t}$$
(3.36)

Here,  $H_{CFIR}(f)$  is the transfer function of the CFIR filter. Since the coefficients of the CFIR filter are symmetric,  $c_p = c_{-p}$ ,  $H_{CFIR}(f)$  is an even and real function. An even and real transfer function does not cause phase shifts in the input-output relationship.  $c_p$  in Eq. (3.36) are the coefficients of the truncated Fourier series for  $H_t(f)$  and thus defined by the next equation.

$$c_{p} = \Delta t \int_{-f_{s}/2}^{f_{s}/2} H_{t}(f) \mathrm{e}^{\mathrm{i}2\pi f p \Delta t} df \qquad (3.37a)$$

$$=\Delta t f_t \int_{-1/(2\tilde{f}_t)}^{1/(2\tilde{f}_t)} \widetilde{H}(\tilde{f}) \cos(2\pi \tilde{f} f_t p \Delta t) d\tilde{f}$$
(3.37b)

$$=\frac{f_t}{f_s}\int_{-1/(2\tilde{f}_t)}^{1/(2\tilde{f}_t)}\frac{1}{1+\lambda_2\tilde{f}^4}\cos(2\pi\tilde{f}\frac{f_t}{f_s}p)d\tilde{f}$$
(3.37c)

$$=2\widetilde{f}_{t}\int_{0}^{1/(2\widetilde{f}_{t})}\frac{1}{1+\lambda_{2}\widetilde{f}^{4}}\cos(2\pi\widetilde{f}\ \widetilde{f}_{t}p)d\widetilde{f}$$
(3.37d)

Where  $f_s = \Delta t^{-1}$  and  $\tilde{f}_t = f_t / f_s$  are the sampling frequency and the normalized

target frequency to the sampling frequency.

The normalization of CFIR filter coefficients with respect to  $\tilde{f}_t$  provides independent coefficients from the target and sampling frequencies.

$$\widetilde{c}_{p} = 2 \int_{0}^{1/(2\widetilde{f}_{t})} \frac{1}{1 + \lambda_{2}\widetilde{f}^{4}} \cos(2\pi\widetilde{f} \ \widetilde{p}) d\widetilde{f}$$
(3.38)

Here,  $\tilde{c}_p = c_p / \tilde{f}_t$  and  $\tilde{p} = \tilde{f}_t p$  are the normalized CFIR filter coefficients and the normalized p. The coefficients  $\tilde{c}_p$  have same values no matter which the sampling and target frequencies are. Fig. (3.8a) shows the normalized coefficients of the CFIR acceleration filter.

To select the truncation number k of Eq. (3.36), the Gibbs phenomenon must be considered. The Gibbs phenomenon explains the rippling on the transfer function of the truncated Fourier series [Hamming 1989]. By the phenomenon, the very outside of the coefficients  $c_k = c_{-k}$  should be near zero to stabilize rippling of the transfer function. Fig. (3.8b) shows zero-crossing points  $p_0$ . Note that  $k = p_0 / \tilde{f}_t$  and the filter size of the CFIR acceleration filter is 2k + 1. Figs. (3.9a, b and c) show the transfer functions of the acceleration filter using CFIR discretization for various filter sizes.  $p_0 = 0.222$  does not have a exact pass band and thus  $p_0 = 0.519$  is the minimum filter size. The filter size  $p_0 = 0.519$ ,  $p_0 = 0.815$  and  $p_0 = 1.112$  are reasonable filter sizes with the



Figure 3.8 Normalized coefficients of the CFIR acceleration filter (a) small scale (b) detail in a large scale



acceptable pass bands and cut-off bands. Since the error of the pass band and the rippling of the cut-off band are reduced as the filter size increases,  $p_0 = 0.519$ ,  $p_0 = 0.815$  and  $p_0 = 1.112$  are recommended as the short, standard and long filter sizes, respectively.

The filtered displacement on the discretized time is defined as the next equation.

$$\widetilde{u}_{t} = \widetilde{f}_{t} \sum_{p=p_{0}\widetilde{f}_{t}}^{p_{0}\widetilde{f}_{t}} \widetilde{c}_{p} \overline{u} (t+p\Delta t)$$
(3.39)

If the truncation number of the Fourier series  $k = p_0 \tilde{f}_t$  is not a natural number, the nearest natural number is the truncation number.

The acceleration filter is also realized by the discretization of the governing equation (Eq. 3.18a) using FDM. The scheme is named by the FDM-FIR filter because the final form of the discretization is an FIR filter. The FDM-FIR filter is totally handled on the time domain, while the CFIR filter is designed on the frequency domain. The CFIR filter design has the forward and inverse Fourier transform processes, which cause lots of numerical error. The FDM-FIR filter, however, does not include any domain shift processes, and gives a more stable and accurate filter than the CFIR filter.

A fourth-order FDM matrix,  $\mathbf{L}_4$ , is utilized during the discretization of the governing equation of the acceleration filter (Eq. 3.18a).

$$\mathbf{L}_{4} = \frac{1}{\Delta t^{4}} \begin{bmatrix} 1 & -4 & 6 & -4 & 1 \\ 1 & -4 & 6 & -4 & 1 \\ & & & \ddots \\ & & & & \ddots \\ & & & & & 1 & -4 & 6 & -4 & 1 \end{bmatrix}$$
(3.40)

Note that  $\mathbf{L}_4$  is not a square matrix; thus, the discretized governing equation requires two initial values and two final values.

$$\overline{\mathbf{u}} = \widetilde{\mathbf{u}} + \lambda_t \begin{bmatrix} \mathbf{L}_4^i & \mathbf{L}_4^s & \mathbf{L}_4^f \end{bmatrix} \begin{bmatrix} \widetilde{\mathbf{u}}^i \\ \widetilde{\mathbf{u}} \\ \widetilde{\mathbf{u}}^f \end{bmatrix}$$
(3.41)

In this equation,  $\mathbf{L}_{4}^{i}$ ,  $\mathbf{L}_{4}^{f}$ ,  $\mathbf{L}_{4}^{s}$ ,  $\mathbf{\overline{u}}$ ,  $\mathbf{\widetilde{u}}$ ,  $\mathbf{\widetilde{u}}^{i}$  and  $\mathbf{\widetilde{u}}^{f}$  are the first two column of  $\mathbf{L}_{4}$ , the last two column of  $\mathbf{L}_{4}$ , the square part of  $\mathbf{L}_{4}$ , the measured displacements, the filtered displacements, the two initial values of the regularized displacement and two final values of the filtered displacement, respectively. Two initial and two final values are required to solve Eq. (3.41).

$$(\mathbf{I} + \lambda_t \mathbf{L}_4^s)^{-1} (\overline{\mathbf{u}} - \lambda_t \mathbf{L}_4^i \widetilde{\mathbf{u}}^i - \lambda_t \mathbf{L}_4^f \widetilde{\mathbf{u}}^f) = \widetilde{\mathbf{u}}$$
(3.42)

The above equation gives filtered displacements using the acceleration filter. The filtered displacements are not disturbed by the noise amplification of the relationship between the displacements and the accelerations.

Eq. (3.42) requires two initial and two final values, but the boundaries of the

time series data are usually unknown. When the boundary information is not exact, effects of the boundary value errors are derived from Eq. (3.42).

$$-\lambda_t (\mathbf{I} + \lambda_t \mathbf{L}_4^s)^{-1} (\mathbf{L}_4^i \widetilde{\mathbf{u}}_{\text{error}}^i + \mathbf{L}_4^f \widetilde{\mathbf{u}}_{\text{error}}^f) = \widetilde{\mathbf{u}}_{\text{boundery error}}$$
(3.43)

The center element of  $\mathbf{u}_{boundery\,error}$  is nearly zero when the data length is long enough. Because  $(\mathbf{L}_4^i \widetilde{\mathbf{u}}_{error}^i + \mathbf{L}_4^f \widetilde{\mathbf{u}}_{error}^f)$  is a zero vector except for the first and last two elements, and  $(\mathbf{I} + \lambda_r \mathbf{L}_4^s)^{-1}$  is a almost banded matrix. The time window technique has been used to solve this type of boundary problem [Hong 2010, Hong 2013]. With the time window technique, Eq. (3.42) is applied to pieces of data, and the center value is picked up in each piece (Fig. 3.10). The window refers to an individual piece. Boundary error effects are negligible when the time window size is long enough, as the center value of the  $\widetilde{\mathbf{u}}_{boundery\,error}$  is nearly zero. The time window technique serves as a finite impulse response (FIR) filter formed by the center row of  $(\mathbf{I} + \lambda_r \mathbf{L}_4^s)^{-1}$ . The FIR filter is applied in the next equation.

$$\widetilde{u}_{t} = \widetilde{u}(t) = \sum_{p=-k}^{k} c_{p} \overline{u}_{t+p} = \sum_{p=-k}^{k} c_{p} \overline{u}(t+p\Delta t)$$
(3.44)

Here,  $\tilde{u}_n$ ,  $\bar{u}_n$ ,  $c_n$  and k are the n<sup>th</sup> step of the output (filtered) displacements, the n<sup>th</sup> step of the input (measured / output from the energy filter) displacements,



Figure 3.10 Concept of overwrapping time window technique



Figure 3.11 coefficients of FDM-FIR and CFIR filters

the FDM-FIR filter coefficients (the center row of  $(\mathbf{I} + \lambda_t \mathbf{L}_4^s)^{-1}$ ) and the window size coefficient, respectively. The window size  $N_w$  is defined by the window size coefficient k.

$$N_w = 2k + 1$$
 (3.45)

Since the very outside of the coefficients  $c_p$  is always near zero, the window size (or the size of the FDM-FIR filter) can be chosen by any values, while the size of the CFIR filter must have specific values. However, it is figured out empirically that the FDM-FIR filter converges on the analytic transfer function faster when the window size is identical to the filter size of the CFIR filter. Accordingly, the truncation number of CFIR filter  $k = p_0 / \tilde{f}_t$  can be referred as the window size of FDM-FIR filter.

$$N_w = 2k + 1 = 2\frac{p}{\tilde{f}_t} + 1 \tag{3.46}$$

 $p \ge 1.112$ ,  $p \ge 0.815$  and  $p \ge 0.519$  are suggested for the recommended, standard and minimum window sizes. The values 1.112, 0.815 and 0.519 are recommended  $p_0$  of CFIR filter. The coefficients of FDM-FIR and CFIR filters is mostly same to each other when  $p = p_0$  (Fig. 3.11). The transfer functions of the FDM-FIR filters, however, are more stable than the transfer functions of the CFIR filters (Figs. 3.12a, b and c).

 $N_w$  of Eq. (3.46) is the window size in the number of steps, but the window size in the time dimension is easier to understand than  $N_w$ 

$$T_{w} = \Delta t (N_{w} - 1) = 2 \frac{p_{0}}{f_{t}} = 2 p_{0} T_{t}$$
(3.46)

Where  $T_w$  and  $T_t = 1/f_t$  are the window size on the time dimension and the target period.  $T_w > 2.2T_t$ ,  $T_w > 1.6T_t$  and  $T_w > 1.2T_t$  are suggested for the recommended, standard and minimum window sized  $T_w$ .

#### 3.3.2 Discretization of Energy Filter

The variation of the object function for the energy filter (Eq. 3.21) is easily discretized by the FEM because it is similar to the variation of the equilibrium equation for the linear elastic continuum.

$$\sum_{e} \int_{V^{e}} \mathbf{N}^{T} \mathbf{N} dV^{e} \overline{\mathbf{U}} = \sum_{e} \int_{V^{e}} \mathbf{N}^{T} \mathbf{N} dV^{e} \widetilde{\mathbf{U}} + \lambda_{s} \sum_{e} \int_{V^{e}} \mathbf{B}^{T} \mathbf{D} \mathbf{B} dV^{e} \widetilde{\mathbf{U}}$$
(3.47)

Here, **N**, **B**,  $\overline{\mathbf{U}}$ ,  $\widetilde{\mathbf{U}}$ ,  $V^e$  and  $\sum_{e}()$  are the shape function matrix, the first derivation of the shape function, the global vector of the input (measured / output from the acceleration filter) displacements, the global vector of the output (filtered)



(a) small scale (b) detail of target frequency (c) detail of cut-off band

displacements (unknown), the domain of each element and the structural compatibility summation, respectively. The left term and the first term on the right are identical to the mass matrix of the FEM for the target continuum with the unit mass density. The second term on the right is identical to the stiffness matrix of the target continuum.

$$\mathbf{M}_{\text{unit}} \overline{\mathbf{U}} = \mathbf{M}_{\text{unit}} \widetilde{\mathbf{U}} + \lambda_s \mathbf{K} \widetilde{\mathbf{U}}$$
(3.48)

In this equation,  $\mathbf{M}_{unit}$  and  $\mathbf{K}$  are the mass matrix of the target continuum with unit mass density and the stiffness matrix of the target continuum, respectively. The matrixes are identical to the matrixes used in forward analyses, except for the mass density. The filtered displacement is determined by solving Eq. (3.48).

$$(\mathbf{M}_{unit} + \lambda_s \mathbf{K})^{-1} \mathbf{M}_{unit} \overline{\mathbf{U}} = \widetilde{\mathbf{U}}$$
(3.49)

The stiffness matrix  $\mathbf{K}$  includes the elastic modulus of each element, i.e., the unknown parameters of the EEE. To solve Eq. (3.49), the unknown values must be sourced from prior information. This prior information does not have to be highly accurate because it has low sensitivity with regard to the final solution of the inverse analysis (Eq. 2.13). If the prior is wholly unknown, a preliminary trial solution of the inverse analysis can serve as the prior.

# 3.4 Acceleration-Energy Filter as Signal Processing

## 3.4.1 Acceleration filter as a signal processing

The acceleration filter is designed by the regularity condition and is realized by a low-pass FIR filter. This is feasible because the object function of the acceleration filter (Eq. 3.16) is designed for suppressing the noise amplification of the second differentiation since it is known that differentiation amplifies highfrequency noises. The acceleration filter is a type of the noise suppression filter to stabilize the second derivatives of noisy measured displacements. Usual noise suppression filters may be available for solving the inverse analysis using EEE. However, the acceleration filter has two advantages: consistency with the energy filter and optimality of the inverse analysis for the continuum. The acceleration filter and the energy filter are applied together for solving inverse analysis of the linear elastic continuum, and these process require consistency of the relationship between the elastic waves and the temporal wave. Without the consistency, the temporal and spatial derivatives have different information. These differences are present serious noise. The acceleration filter contains the physical meaning of the equilibrium equation and its FIR form has the same physical meaning as well. The physical meaning is the media of the consistency between the acceleration and energy filters.

### 3.4.2 Energy filter as a Signal Processing

The final form of the energy filter is a spatial filter in three-dimensional space. For the rectangular-meshed FEM, the energy filter takes precisely the same form as a spatial low-pass FIR filter, which is usually used in image processes. A lowpass FIR filter may work for the inverse analysis for the continuum if it is designed properly. However, the energy regularization has certain specialties which make it capable of solving the inverse analysis of the linear elastic continuum.

The first advantage is the boundary conditions. The boundary conditions of the energy filter is identical to the boundary conditions of the equilibrium equation of the continuum. This makes the solutions around the boundaries trustable while the other spatial filters do not give proper solutions around the boundaries. For example, the spatial filter using a  $L_2$ -norm spatial derivative regularity condition exists, as commonly used by other researchers [Park 2006, Park 2009]. The  $L_2$ -norm type spatial filter employs the spatial derivatives of the displacement with a  $L_2$ -norm error function.

$$\min_{\widetilde{u}} \Pi = \frac{1}{2} \int_{V} (\overline{u} - \widetilde{u})^{2} dV + \frac{\lambda_{s}}{2} \int_{V} \left\| \nabla \widetilde{u} \right\|_{2}^{2} dV$$
(3.50)

The governing equation (Eq. 3.51a) and the boundary condition (Eq. 3.51b) are derived by the same procedure used for acceleration filter.

$$0 = \widetilde{u} - \overline{u} - \lambda_s \nabla^2 \widetilde{u} \tag{3.51a}$$

$$\delta \tilde{u} \nabla \tilde{u} n \Big|_{s} = 0 \tag{3.51b}$$

The boundary condition of the  $L_2$ -norm spatial derivative filter is not zero on the traction-free boundaries of the continuum. A difference between the boundary condition of the continuum and the  $L_2$ -norm spatial derivative filter has detrimental effects on solutions around the traction-free boundaries. On the other hand, The boundary conditions of the energy filter is identical to the boundary conditions of the continuum, and the energy filter works properly on any type of boundary.

The second is information about connectivity between nodes. The FIR form of the spatial filters can consider only well-arranged nodes, rectangular FEM elements. However, FEM can employ complicated elements to discretize the continuum and cannot be handled by FIR spatial filters. However, the energy filter contains information about the connectivity between nodes, as is the case with the discretized continuum. For this reason, the energy filter can handle any type of meshes.

The final advantage is the physical meaning for consistency. This was discussed in section 3.4.1.

# 3.5 Bias Compensation for EEE Using Filtered Displacement

The reconstruction of the bias terms, Eqs. (2.23a and b), are based on the white noise assumption. Noise in the filtered displacement, however, is not white noise but colored noise, because filtered white noise changes into colored noise. Moreover, since the noise undergoes the energy filter, the colored noise is not independent from each other. Thus, covariances between colored noise must be considered for the bias compensation.

The sequence below concerns the covariances of the filtered noise by fourdimensional filters, which are the synthesized filters of the acceleration filter and the energy filters. This step gives the covariances of Eq. (2.22c). White noise w(x, y, z, t) has the position of (x, y, z) with variance  $\sigma^2(x, y, z)$ . The white noise after undergoing four-dimensional filtering  $h_k(x, y, z, t)$  or  $h_l(x, y, z, t)$ becomes the colored noise  $c_k$  or  $c_l$ , respectively.

$$c_k(x, y, z, t) = \sum_{\xi} \sum_{\psi} \sum_{\zeta} \sum_{\tau} h_k(\xi, \psi, \zeta, \tau) w(x - \xi, y - \psi, z - \zeta, t - \tau)$$
(3.52a)

$$c_l(x, y, z, t) = \sum_{\xi} \sum_{\Psi} \sum_{\zeta} \sum_{\tau} h_l(\xi, \Psi, \zeta, \tau) w(x - \xi, y - \Psi, z - \zeta, t - \tau)$$
(3.52b)

The covariance of the two types of colored noise is then  $\sigma_{k,l}$  and is derived by the equation below. The colored noise  $c_k(x_i, y_i, z_i, t)$  is on the i<sup>th</sup> node and undergoes  $h_k$ . The colored noise  $c_l(x_j, y_j, z_j, t)$  is on the j<sup>th</sup> node and undergoes  $h_l$ . Orthogonality between each instance of white noise is assumed.

$$\sigma_{k,l}^{2}(x_{i}, y_{i}, z_{i}, x_{j}, y_{j}, z_{j}) = \frac{1}{n_{t}} \sum_{t} [c_{k}(x_{i}, y_{i}, z_{i}, t) \times c_{l}(x_{j}, y_{j}, z_{j}, t)]$$
(3.53a)

$$= \frac{1}{n_{t}} \sum_{\tau} \left\{ \begin{bmatrix} \sum_{\xi} \sum_{\psi} \sum_{\zeta} \sum_{\tau} h_{k}(\xi, \psi, \zeta, \tau) w(x_{i} - \xi, y_{i} - \psi, z_{i} - \zeta, t - \tau) \end{bmatrix} \\ \times \begin{bmatrix} \sum_{\xi} \sum_{\psi} \sum_{\zeta} \sum_{\tau} h_{i}(\xi, \psi, \zeta, \tau) w(x_{j} - \xi, y_{j} - \psi, z_{j} - \zeta, t - \tau) \end{bmatrix} \right\}$$
(3.53b)

$$= \frac{1}{n_{t}} \sum_{t} \left\{ \begin{bmatrix} \sum_{\xi} \sum_{\psi} \sum_{\zeta} \sum_{\tau} h_{k}(x_{i} - \xi, y_{i} - \psi, z_{i} - \zeta, \tau) w(\xi, \psi, \zeta, t - \tau) \end{bmatrix} \\ \times \begin{bmatrix} \sum_{\xi} \sum_{\psi} \sum_{\zeta} \sum_{\tau} h_{l}(x_{j} - \xi, y_{j} - \psi, z_{j} - \zeta, \tau) w(\xi, \psi, \zeta, t - \tau) \end{bmatrix} \right\}$$
(3.53c)

$$= \frac{1}{n_t} \sum_{t} \left\{ \sum_{\xi} \sum_{\psi} \sum_{\zeta} \sum_{\tau} \left[ \frac{h_k(x_i - \xi, y_i - \psi, z_i - \zeta, \tau) w(\xi, \psi, \zeta, t - \tau)}{\kappa h_l(x_j - \xi, y_j - \psi, z_j - \zeta, \tau) w(\xi, \psi, \zeta, t - \tau)} \right] \right\}$$
(3.53d)

$$=\sum_{\xi}\sum_{\Psi}\sum_{\zeta}\sum_{\tau}\left\{ \begin{aligned} h_{k}(x_{i}-\xi,y_{i}-\Psi,z_{i}-\zeta,\tau)h_{l}(x_{j}-\xi,y_{j}-\Psi,z_{j}-\zeta,\tau) \\ \times \left[\frac{1}{n_{t}}\sum_{\iota}w(\xi,\Psi,\zeta,t-\tau)w(\xi,\Psi,\zeta,t-\tau)\right] \end{aligned} \right\}$$
(3.53e)

$$= \sum_{\xi} \sum_{\Psi} \sum_{\zeta} \sum_{\tau} \left[ \begin{array}{c} h_k(x_i - \xi, y_i - \Psi, z_i - \zeta, \tau) \\ \times h_l(x_j - \xi, y_j - \Psi, z_j - \zeta, \tau) \sigma^2(\xi, \Psi, \zeta) \end{array} \right]$$
(3.53f)

$$= \sum_{\xi} \sum_{\Psi} \sum_{\zeta} \sigma^{2}(\xi, \Psi, \zeta) \sum_{\tau} \begin{bmatrix} h_{k}(x_{i} - \xi, y_{i} - \Psi, z_{i} - \zeta, \tau) \\ h_{l}(x_{j} - \xi, y_{j} - \Psi, z_{j} - \zeta, \tau) \end{bmatrix}$$
(3.53g)

The orthogonality between each white noise instance is used for Eq. (3.53c). With the Eqs. (2.22c) and (3.53g), the bias terms of Eq. (2.21) are reconstructed with the variances of the noise, the four-dimensional acceleration-energy filters, and the FEM models of the continuum.

$$\overline{\mathbf{Q}}_{\text{bias}}(i,j) = n_t \left\{ \sum_n \sum_m q_{nm}^{ij} \sum_{\xi} \sum_{\Psi} \sum_{\zeta} \sigma^2(\xi,\Psi,\zeta) \sum_{\tau} \begin{bmatrix} h_{disp}(x_i - \xi, y_i - \Psi, z_i - \zeta, \tau) \\ \times h_{disp}(x_j - \xi, y_j - \Psi, z_i - \zeta, \tau) \end{bmatrix} \right\} \quad (3.54a)$$

$$\overline{\mathbf{M}}_{\text{bias}}(i) = n_t \left\{ \sum_j \sum_n \sum_m m_{nm}^{ij} \sum_{\xi} \sum_{\Psi} \sum_{\zeta} \sigma^2(\xi,\Psi,\zeta) \sum_{\tau} \begin{bmatrix} h_{disp}(x_i - \xi, y_i - \Psi, z_i - \zeta, \tau) \\ \times h_{acc}(x_j - \xi, y_j - \Psi, z_i - \zeta, \tau) \end{bmatrix} \right\} \quad (3.54b)$$

Here,  $\sigma^2(x, y, z)$ ,  $q_{m,n}^{i,j}$ ,  $m_{m,n}^{i,j}$ ,  $h_{disp}$  and  $h_{acc}$  are the covariance values of white noise for the measuring point on the (x, y, z), the (m, n) element of the matrix  $\mathbf{q}^{i^T}\mathbf{q}^j$ , the (m, n) element of the matrix  $\mathbf{q}^{i^T}\mathbf{m}^j$ , the filter for the displacement of EEE and the filter for the acceleration of EEE, respectively. In the other words,  $h_{disp}$  is the synthesized filter of the acceleration and the energy filter.  $h_{acc}$  can be understood as the synthesized filter of the acceleration filter, the energy filter and the central FDM for second order derivatives using second order accuracy, as well.

 $\overline{\mathbf{Q}}_{\text{bias}}$  and  $\overline{\mathbf{M}}_{\text{bias}}$  are substituted into  $\mathbf{Q}_{\text{bias}}$  and  $\mathbf{M}_{\text{bias}}$  of Eq. (2.21), and the biases in the solutions are eliminated.

# 4. Example and Application

An example and an application are introduced in this chapter. The example is a linear elastic continuum problem using dynamic displacements under the plane stress condition. The example uses the schemes in chapter 3. The application is medical imaging with ultrasound equipment. Human tissues are under an incompressible condition in this case and are governed by the Helmholtz equation. The Helmholtz equation is a continuum in the incompressible condition. The inverse analysis for the Helmholtz equation is slightly different from the inverse analysis for the continuum. This is introduced in this chapter.

# 4.1 Example: Aluminum Plate

A thin aluminum plate model (Fig. 4.1) is employed to demonstrate the efficiency of the acceleration-energy filter and the bias compensation. Aluminum plates are used in many fields, especially with regard to machines and vehicles. In this case, the plate is modeled by a plane stress FEM model. Newmark's method is applied to a dynamic analysis. The numerical calculations are done by MATLAB.

In the first, the efficiency of the schemes is assessed in the ideal condition, which is satisfies the assumptions in Section 2.1 with assumptions of a long enough measuring time and a perfect choice of the target frequency. After that, some



Figure 4.1 The aluminum plate example

unideal conditions are verified. The unideal conditions are the small window sizes, the short measuring times, the damping of structures, the large noise levels, the not proper target frequencies, the bias compensation with unknown noise variances and the noise of specific frequencies.

### 4.1.1 Ideal Example

The specifications of the example are given in Tab. (4.1). The variances of the white noises are assumed to be known values. The target frequency, target accuracy and window size must be chosen to apply the acceleration filter. To define the target frequency, measured displacement must be analyzed in the frequency domain. The measured displacement on the point (A) of Fig. (4.1) is displayed in Fig. (4.2) (a piece of the time line). The frequency spectrum of the displacements is given in Fig. (4.3). The target frequency is chosen according to the highest dominant frequency, 5.15kHz. The target accuracy is defined as 0.97 by an engineering sense. The window size is defined as 45 steps, which is a recommended window size.

Material	Aluminum ( $E = 70GPa$ , $\rho = 2.7Mg/m^3$ )	
Size	1m X 1m X 0.1m	
Load	Sudden release of unit distributed load &	
	Load free during the measurement time	
Boundaries	1 fixed end, 3 traction free ends	
Inclusion	Hard: 140GPa	
	Soft: 35GPa	
Modeling	50X50 Q4 FEM, plane stress	
Measurement type	Displacement	
Sampling rate	100kHz	
Measurement time	0.14s (14000steps)	
Noise	5% proportional white noise	
	(Signal to noise ratio: 30.8dB)	

Table 4.1Specification of the aluminum example



Figure 4.3 Frequency spectrum and the transfer function of acceleration filter (the aluminum example)

Figs. (4.4a and b) show the results for hard and soft inclusion cases, respectively. The hard and soft inclusions are detected. Fig. (4.5a and b) show the results of the line (B-B') in Fig. (4.1). This figure shows the effectiveness of the energy filter and the bias compensation. The acceleration filter is applied to all of results in the figures because results without the acceleration filter are nearly identical to meaningless white noise. The figures show that the acceleration-energy filter and the bias compensation make the results accurate.

To compare the energy filter and the regularization schemes, the results of the regularization are also displayed in Fig. (4.5a and b). The acceleration filter is applied to the regularization because low-pass filters are generally used in the temporal dimensions. The L<sub>2</sub>-norm Tikhonov regularization is applied with GMS [Lee 1999, Park 2001]. Prior information of the material properties is given by 70GPa, which is the Young's modulus of aluminum. As explained in chapter 2, the regularization scheme improves the results of soft inclusion cases. A side effect, low accuracy of the inclusion, is well known in the references and is thus ignorable. However, a side effect of the hard inclusion case is too severe to be ignored. On the other hand, the energy filter works properly in both cases.

Figs. (4.4 a and b) and (4.5a and b) show that the stiffness discontinuous surfaces are smoothed. It is caused by the energy filter. The inverse analysis for the continuum uses not information of displacement but information of strain. It is clear because the equilibrium equation (Eq. 2.1) is consist of a strain term. In the stiffness discontinuous surfaces, the strain functions are step functions, and the









Figure 4.6 Effect of energy filter on the stiffness discontinuous surface

step functions are fractional functions in the frequency domain. When the energy filter, which is a low-pass filter, is applied to the measured displacement, most of the stiffness discontinuity information is passed through but the tails of the fractional functions are filtered out (Fig. 4.6). This sequence cuts off the high frequency components of the step functions, and the step functions change into smoothed functions.

### 4.1.2 Verification 1: Window Size

In the ideal example, the recommended window size is employed. In some situations, however, the standard window size or the minimum window size might be chosen because the filter with the bigger window size losses more data than the filter with the smaller window size. Fig. (4.7) shows results using the recommended, standard and minimum window sizes. The results using various



Figure 4.7 Results with various window sizes (aluminum example)

window sizes are almost same to each others. Since the other conditions except the window size are ideal, any window sizes can be employed for the acceleration filter. In some unideal condition, however, the recommended window size gives the more accurate result.

### 4.1.3 Verification 2: Measuring Time

The ideal case uses very long enough measuring time, which provides high quality information. However, the long enough measuring time is not allowed in some applications, especially applications with damping. The inverse analysis using EEE in this thesis uses an assumption that damping can be ignored. The damping is ignorable since relatively few vibrations are included in the measuring time (Section 2.1). To verify effects of the short measuring time, 2.2waves, 0.9waves and 0.4waves of the longest wave length is applied for the measuring time. The minimum window size is applied for reducing loss of data. Fig. (4.8) shows the results. Results of the inclusion is less accurate than the ideal case since lot s of information is truncated by the short measuring time. however, the results are acceptable because the inclusion is well detected in the all cases.



Figure 4.8 Results with various measuring times (aluminum example)

### 4.1.4 Verification 3: Damping

The damping of structures is ignored by the assumption in Section 2.1, but most of real structures have damping. To verify effects of the assumption, 1%, 5% and 9% damping using Rayleigh damping is applied for the forward analysis. The inverse analysis is done with the 0% damping assumption. Since only few vibrations must be included in the measurement time, only a wave of the longest wave length is chosen for the measuring time. For the inverse analyses of the damping cases, a different target frequency is applied since the frequency characteristics of the damped system are different from it of the ideal case (Fig. 4.9). The target frequency for the damping cases is chosen by 2.3kHz. Fig. (4.10) shows the results. The stiffness discontinuous surfaces are more smoothed than the ideal case because the smaller target frequency gives a more strict spatial filter. The damping acts as a type of noise in the measured displacement but is not considered by the bias compensation. For this reason, the bias compensation gives few error in the damping cases. The inverse analysis using damping term [Park 2007] might help to improve the result. It considers a Rayleigh damping term for identifying system parameters.



Figure 4.9 Selection of target frequency in damping case (aluminum example)



Figure 4.10 Results with various damping (aluminum example)

### 4.1.5 Verification 4: Noise Level

Most of cases, proportional noise levels are lower than 5%. However, an extremely high level of noise might exist in some cases. For verifying effects of high level noises, 20%, 30% and 50% of proportional noises are applied to the measured displacement. The relationships between the proportional error, the signal to noise ratio (SNR) and the signal to noise ratio in decibel (SNR<sub>dB</sub>) are presented in Tab. (4.2).

$$SNR = \sigma_s^2 / \sigma_n^2 \tag{4.1}$$

$$SNR_{dB} = 10\log_{10}(SNR) \tag{4.2}$$

Fig. (4.11) shows the results. By the results, the acceleration-energy filter and the bias compensation are work properly for up to the 30% of proportional error.

Table 4.2 Relationship between proportional error and signal to holse ratio		
Proportional error	SNR	$SNR_{dB}$
5%	1189.8	30.8
20%	74.3	18.7
30%	33.1	15.2
50%	11.9	10.8

 Table 4.2
 Relationship between proportional error and signal to noise ratio



Figure 4.11 Results with various noise levels (aluminum example)



Figure 4.12 Results using various target frequencies (aluminum example)
# 4.1.6 Verification 5: Target Frequency

Since the target frequency of the acceleration-energy filter is pre-defined by an engineering sense, a not proper target frequency might be chosen by an incorrect engineering sense. For verifying the effects of unideal target frequency, the target frequencies  $f_t = (5.15 \times 1.5)kHz$ ,  $f_t = (5.15 \times 2)Hz$  and  $f_t = (5.15 \times 3)Hz$  is applied to the acceleration-energy filter of the aluminum plate example. The ideal example uses the target frequency  $f_t = 5.15 kHz$ . Fig. (4.12) shows results using the various target frequencies. Even though the target frequency is not the best, the inclusion is identified by the inverse analysis using the acceleration-energy filter and the bias compensation. The bias of the solution is not perfectly eliminated by bias compensation because the third assumption of the bias compensation is not satisfied. The third assumption of the bias compensation is that the measured data is long enough and the bias terms are converged enough. The signal undergoing the filter using a bigger target frequency have more noise than the signal undergoing the filter using the ideal target frequency. The bias terms of the signal with more noise is converged slower than it of the ideal one.

#### 4.1.7 Verification 6: Bias Compensation without Noise Variance Information

The bias compensation using Eqs. (3.54 a and b) and Eq. (2.21) requires the variance of the noise in the measured displacement. However, the variance of the noise is usually unknown. To remove the bias of the solution without the noise variance, Eq. (2.24) must be employed. As mentioned in Section 2.4, Eq. (2.24) has too much unknowns and additional information can reduce the number of the unknown. A proportional noise assumption is employed for reducing the number of the unknown and then, Eq. (2.24) changes into the function of a noise level.

$$\min \Pi(\mathbf{nl}) = \left\| \overline{u}_t - u_t(E_b(\mathbf{nl})) \right\|_2^2 \tag{4.3}$$

Where nl is the noise level in %. Fig. (4.13a) shows the normalized means square errors of Eq. (4.3) which is normalized by the square of the maximum displacement.  $u_t(E_b(nl))$  using nl < 3% or nl > 7% cannot be analyzed because the solutions the inverse analyses  $E_b(nl)$  using nl < 3% or nl > 7% are not reasonable (negative or too big values). By Fig. (4.13a), 5.5% of noise level is the optimal solution of Eq. (4.3). Note that the exact noise level is 5%. The solutions using the exact noise level (5%) and the estimated noise level (5.5%) are represented in Fig. (4.13b). The solution using the estimated noise level is acceptable.



Figure 4.13 Results when noise variance is unknown (aluminum example) (a) Normalized mean square error (b) results of inverse analyses

# 4.1.8 Verification 7: Noise of Specific Frequency

Specific frequency noise can be included in the measured displacement. Four types of the specific frequency noise is employed to verify the noise. The first noise is a single wave noise. The noise has 20kHz frequency, which is higher than the target frequency, and an amplitude of the maximum displacement. The second noise is a single wave noise with a frequency which is lower than the target frequency. The noise has 2kHz frequency and the same amplitude to the first one. The third noise is static noise which is constant for all nodes. The constant value is the maximum of the displacement. The final one is static noise which is different values for all nodes. The static noise is generated by the random value which is uniformly distributed between the maximum displacement and the negative value of it. Fig. (4.14a) shows the solution without additional denoising schemes. The case of the 20kHz noise find out the inclusion since the 20kHz noise, which is higher than the target frequency, is filtered out by the acceleration The bias compensation, however, does not work properly because the white filter. noise assumption is not satisfied. The case of the constant static noise also find out the inclusion since the constant static noise does not effect to the strain, which is actually used by the inverse analysis. The bias compensation, however, does not work properly by the same reason of the 20kHz noise case. The solutions of the other cases are meaningless.

For these cases, denoising schemes for the specific noise must be employed. The static noise is a kind of the single frequency noise with OHz. In this example,



Figure 4.14 Results with specific frequency noise (aluminum example) (a) W/O proper denoising (b) with proper denoising

the adaptive line enhancer (ALE) is employed because ALE is very effective to remove the single frequency noise [Haykin 2002]. The solutions using ALE are shown in Fig. (4.14b). the results are almost identical to the solution of the ideal example.

# 4.2 Application to Medical Imaging: Ultrasonic Elastography

The inverse analysis using EEE can be applied to medical imaging to find tumors and cancers in human bodies. The inverse analysis using EEE works well with medical imaging because the real-time processes make diagnoses exact and reduce the costs. The acceleration-energy filter and the bias-compensation must also be applied with the inverse analysis. Both civil structures and human bodies can be modeled by continua, showing the same behaviors apart from the material properties. In this section, ultrasonic elastography is introduced. The inverse analysis using the Helmholtz equation related to this process are also introduced.

# 4.2.1 Introduction

Ultrasonic elastography is a type of medical imaging which reconstructs and visualizes the material properties of human organs. The elasticity of tumors/cancers differs from that in normal tissue samples. Ultrasonic elastography has fewer side effects than other tests used to find tumors, such as MRI, radiation tests and biopsies. The inverse analysis using EEE is a real-time or near-real-time scheme. The inverse analysis using EEE has higher accuracy



Figure 4.15 Ultrasonic measuring equipments (a) left: picture of the equipments (b) right: concept of the equipments

than x-rays and a lower cost compared to MRI. Here, 'cost' refers to both the time and money.

Ultrasonic elastography uses one-directional displacement in a 2D space, as the ultrasonic measuring equipment (Fig. 4.15a) is a series of 1D probes. This equipment measures the displacements in the z-direction on the x-z plane, as shown in Fig. (4.15b).

The Helmholtz equation is the best for analyzing responses of the human body. The Helmholtz equation is an incompressible case of the continuum equation. A human body can be assumed as an incompressible continuum, as it is filled with water, which is almost incompressible.

A transient shear wave load is employed. This is the latest scheme used to apply loads onto tissues. A uniformly distributed load by ultrasonic is released at once. This method has two advantages. The first is that external forces do not exist during the measurement. The second advantage is that concentrated loads can be generated deep inside of bodies by the ultrasonic equipment, allowing tumors deep inside the body to be detected.

The boundaries are assumed to be infinite boundaries, as the boundary conditions of the bodies are unknown. This assumption is realized by a short measuring time which is shorter than that for the wave to reach the boundary.

# 4.2.2 Inverse Analysis for Helmholtz Equation

The Helmholtz equation is an equation to analyze the incompressible continuum. It is derived from the Navier's equation.

$$(\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) + \mu\nabla^2 \mathbf{u} + \mathbf{b} = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2}$$
(4.4)

according to the incompressible condition,  $\nabla \cdot \mathbf{u}$  is eliminated.

$$\mu \nabla^2 \mathbf{u} + \mathbf{b} = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2}$$
(4.5)

The displacement components are independent from each other and then, the equation can be treated as a one dimensional problem.

$$\mu \nabla^2 u_i + b_i = \rho \frac{\partial^2 u_i}{\partial t^2}$$
(4.6)

The equation above is the Helmholtz equation. The equation below is the variation of the Helmholtz equation.

$$\int_{V} \delta u_i (\mu \nabla^2 u_i + b_i - \rho \ddot{u}_i) dV = 0$$
(4.7)

Integration by parts of the above equation leads to the subsequent equation.

$$-\int_{V} \delta \nabla u_{i} \mu \nabla u_{i} dV + \int_{S} \delta u_{i} \mu \nabla u_{i} \cdot \mathbf{n} dS + \int_{V} \delta u_{i} b_{i} dV - \int_{V} \delta u_{i} \rho \ddot{u}_{i} dV = 0$$
(4.8)

The second term is a boundary condition and is satisfied on the fixed and traction-free boundaries. The third term is the body force term; it is eliminated by the assumption outlined in chapter 2. The equation above is discretized by FEM and applied to the inverse analysis using EEE.

$$\Pi = \sum_{t} \left\| \sum_{e} \mu^{e} \mathbf{q}^{e} \overline{\mathbf{u}}^{e} + \sum_{e} \mathbf{m}^{e} \mathbf{L}_{2} \overline{\mathbf{u}}^{e} \right\|_{2}^{2}$$
(4.9)

Here,  $\overline{\mathbf{u}}^{e}$  is the local measured displacement vector. The unknown parameters are not the Young's modulus but the shear modulus  $\mu$  in the inverse analysis for the Helmholtz equation. In addition,  $\mathbf{q}^{e}$  and  $\mathbf{m}^{e}$  are defined by the equations below.

$$\mathbf{q}^{e} = \int_{V^{e}} \mathbf{B}^{T} \mathbf{B} dV \tag{4.10a}$$

$$\mathbf{m}^{e} = \int_{V^{e}} \rho \mathbf{N}^{T} \mathbf{N} dV \tag{4.10b}$$

Eqs. (4.10a and b) are identical to Eqs. (2.7a and b), which is the inverse analysis for the continuum, and the next sequences are identical to those of the inverse analysis for the continuum. The solutions of Eq. (4.9) are shown next.

$$\frac{\partial \Pi}{\partial \mu^{i}} = \sum_{t=1}^{n} \sum_{e} \mu^{e} (\mathbf{q}^{i} \mathbf{u}^{e})^{T} \mathbf{q}^{e} \overline{\mathbf{u}}^{e} + \sum_{t=1}^{n} \sum_{e} (\mathbf{q}^{i} \mathbf{u}^{e})^{T} \mathbf{m}^{e} \mathbf{L}_{2} \overline{\mathbf{u}}^{e} = 0$$
(4.11)

The matrix form of the above equation is the equation below.

$$\mathbf{M} + \mathbf{Q}\boldsymbol{\mu} = \mathbf{0}$$
(4.12a)  

$$\mathbf{M} = \sum_{t=1}^{n} \begin{bmatrix} \overline{\mathbf{U}}^{T} \mathbf{q}_{t}^{1^{T}} (\mathbf{m}_{t}^{1} + \mathbf{m}_{t}^{2} + \dots + \mathbf{m}_{t}^{e}) \overline{\mathbf{U}} \\ \overline{\mathbf{U}}^{T} \mathbf{q}_{t}^{2^{T}} (\mathbf{m}_{t}^{1} + \mathbf{m}_{t}^{2} + \dots + \mathbf{m}_{t}^{e}) \overline{\mathbf{U}} \\ \vdots \\ \overline{\mathbf{U}}^{T} \mathbf{q}_{t}^{e^{T}} (\mathbf{m}_{t}^{1} + \mathbf{m}_{t}^{2} + \dots + \mathbf{m}_{t}^{e}) \overline{\mathbf{U}} \end{bmatrix}$$
(4.12b)  

$$\mathbf{Q} = \sum_{t=1}^{n} \begin{bmatrix} \overline{\mathbf{U}}^{T} \mathbf{q}_{t}^{1^{T}} \mathbf{q}_{t}^{1} \overline{\mathbf{U}} & \overline{\mathbf{U}}^{T} \mathbf{q}_{t}^{1^{T}} \mathbf{q}_{t}^{2} \overline{\mathbf{U}} & \dots & \overline{\mathbf{U}}^{T} \mathbf{q}_{t}^{1^{T}} \mathbf{q}_{t}^{e} \overline{\mathbf{U}} \\ \overline{\mathbf{U}}^{T} \mathbf{q}_{t}^{2^{T}} \mathbf{q}_{t}^{1} \overline{\mathbf{U}} & \overline{\mathbf{U}}^{T} \mathbf{q}_{t}^{2^{T}} \mathbf{q}_{t}^{2} \overline{\mathbf{U}} & \dots & \overline{\mathbf{U}}^{T} \mathbf{q}_{t}^{2^{T}} \mathbf{q}_{t}^{e} \overline{\mathbf{U}} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{\mathbf{U}}^{T} \mathbf{q}_{t}^{e^{T}} \mathbf{q}_{t}^{1} \overline{\mathbf{U}} & \overline{\mathbf{U}}^{T} \mathbf{q}_{t}^{e^{T}} \mathbf{q}_{t}^{2} \overline{\mathbf{U}} & \dots & \overline{\mathbf{U}}^{T} \mathbf{q}_{t}^{e^{T}} \mathbf{q}_{t}^{e} \overline{\mathbf{U}} \end{bmatrix}$$
(4.12c)

The acceleration filter for the Helmholtz equation is identical to that of the acceleration filter for the continuum.

$$(\mathbf{I} + \lambda_t \mathbf{L}_4^s)^{-1} (\overline{\mathbf{u}} - \lambda_t \mathbf{L}_4^i \mathbf{u}^i - \lambda_t \mathbf{L}_4^f \mathbf{u}^f) = \mathbf{u}$$
(4.13)

The energy filter is identical to it of the continuum, except for  $\mathbf{D}$ .  $\mathbf{D}$  in Eq. (3.47) is replaced by an identity matrix for the Helmholtz equation. The equations next are the energy filter and its matrix form.

$$\sum_{e} \int_{A^{e}} \mathbf{N}^{T} \mathbf{N} t dA^{e} \overline{\mathbf{U}} = \sum_{e} \int_{A^{e}} \mathbf{N}^{T} \mathbf{N} t dA^{e} \widetilde{\mathbf{U}} + \lambda \sum_{e} \int_{A^{e}} \mu_{e} \mathbf{B}^{T} \mathbf{B} t dA^{e} \widetilde{\mathbf{U}}$$
(4.14a)

$$\mathbf{M}_{\text{unit}} \overline{\mathbf{U}} = \mathbf{M}_{\text{unit}} \widetilde{\mathbf{U}} + \lambda_s \mathbf{K} \widetilde{\mathbf{U}}$$
(4.14b)

The reconstructed bias terms of the inverse analysis for the Helmholtz equation are the equation below.

$$\mathbf{Q}_{\text{bias}}(i,j) = n_t \left\{ \sum_n \sum_m q_{nm}^{ij} \sum_{\xi} \sum_{\Psi} \sum_{\zeta} \sigma^2(\xi, \Psi, \zeta) \sum_{\tau} \begin{bmatrix} h_{disp}(x_i - \xi, y_i - \Psi, z_i - \zeta, \tau) \\ \times h_{disp}(x_j - \xi, y_j - \Psi, z_i - \zeta, \tau) \end{bmatrix} \right\}$$
(6.12a)  
$$\overline{\mathbf{M}}_{\text{bias}}(i) = n_t \left\{ \sum_j \sum_n \sum_m m_{nm}^{ij} \sum_{\xi} \sum_{\Psi} \sum_{\zeta} \sigma^2(\xi, \Psi, \zeta) \sum_{\tau} \begin{bmatrix} h_{disp}(x_i - \xi, y_i - \Psi, z_i - \zeta, \tau) \\ \times h_{acc}(x_j - \xi, y_j - \Psi, z_i - \zeta, \tau) \end{bmatrix} \right\}$$
(6.12b)

Above equations are identical to Eqs. (4.54a and b).

# 4.2.3 Human Skin Tissue Example

The example of ultrasonic elastography here is a model of human skin tissue. This model is based on several references [Fink 2004, Park 2006, Park 2009]. The shape of the example is shown in Fig. (4.16), and the specifications are given in Tab. (4.3). The target is a plane stain model with an infinite thickness. The measurement time is defined as 0.01 seconds to satisfy the infinite boundary assumption since the transient wave reaches the boundaries at 0.01 seconds. The variance of the white noise is assumed to have a known value in each case. Newmark's method is applied to dynamic analysis. Numerical calculations are done by MATLAB.

The tree parameters must be defined to use the acceleration-energy filter. Fig. (4.17) shows the measured displacement on the point (A) of Fig. (4.16). The signal is a wave, not a vibration, due to the infinite boundary assumption. Information about the signal is concentrated at a frequency of zero, and the target frequency is defined as 1kHz (Fig. 4.18). The target accuracy is defined as 0.95 and the minimum window size is employed for reducing loss of data.

Material	Human skin ( $\mu = 36kPa$ , $\rho = 1000kg / m^3$ )
Size	$1 \text{m X} 1 \text{m X} \infty$
Load	Sudden release of unit distributed load &
	Load free during the measurement time
Boundaries	1 fixed end, 3 traction free ends
Inclusion	Hard: 72kPa
	Soft: 18GkPa
Modeling	100X100 Q4 FEM, Helmholtz equation, plane strain
Measurement type	Displacement
Sampling rate	100 <i>kHz</i>
Measurement time	0.01s (1000steps)
Noise	5% proportional white noise
	(Signal to noise ratio: 30.8dB)

 Table 4.3
 Specification of the human skin tissue example

The above choices are not in fact the best ones. 500Hz and 0.97 are better choices for the target frequency and target accuracy, respectively. However, the required window size with these values of 500Hz and 0.97 is larger than the number of total sampling points. Note that the number of sampling points lost is identical to the size of the window. To reduce the loss of sampling points, performance of the acceleration-energy filter must be sacrificed. Values of 1kHz, and 0.95 are chosen as the target frequency and the target accuracy, respectively, according to the an engineering sense.

Figs. (4.19a and b) are the reconstructed shear modulus of the hard and soft inclusion cases, respectively. These results of most areas are acceptable, but the results for the center and boundaries are poor, as the information of the center and the boundaries are included during the beginning and the end of the measuring time and are thus lost during the acceleration filtering process. Figs. (4.20a, b, c and d) are the results of the line (B-B') in Fig. (4.16). The results without acceleration filter are not displayed for the same reason given with the aluminum example. These figures show that the energy filter and the bias compensation make the results accurate. Note that the energy filter in this example is more important than in the aluminum example because the performance of the acceleration filter is lower than that in the aluminum example case.



Figure 4.16 The human skin tissue example



Figure 4.18 Frequency spectrum and the transfer function of acceleration filter (the human skin tissue example)





Figure 4.20 Results of human skin example on the line (B-B') (a) hard inclusion case: small scale (b) hard inclusion case: large scale



(c) soft inclusion case: small scale (d) soft inclusion case: large scale

# 5. Conclusion

Solution stabilizers for the inverse analysis using EEE, the acceleration-energy filter and the bias compensation are introduced in this thesis. EEE for the inverse analysis using measured displacement has two types of solution errors: ill-posedness and bias.

Ill-posedness is a well-known problem of the inverse analysis, and the regularization schemes have been used to reduce the ill-posedness. The regularization schemes work well for cases involving soft inclusions. However, efforts of the regularization schemes are not work properly in cases involving hard inclusions. The acceleration-energy filter has been proposed as an alternative. The filter imposes physical conditions of the continuum to the measured displacement. The physical conditions are the finiteness of the strain energy and acceleration. If the measured displacement does not satisfy the physical laws, the solution of the inverse analysis cannot be trusted, as the noise which does not satisfy the physical laws is amplified by the displacement-acceleration relationship and the displacement-strain relationship. The acceleration-energy filter ensures the physical laws of continuum and makes the inverse analysis trustable. The acceleration-energy filter is a temporal-spatial filter with three spatial dimensions and one temporal dimension. The temporal-spatial filter has complicated characteristics and requires much computational effort. To solve this problem, the

acceleration-energy filter is separated into the acceleration filter and the energy filter.

The acceleration filter functions as a low-pass filter. Low-pass filters are generally used to stabilize numerical differentiations. However, acceleration filter has strong physical meaning which satisfies consistency between the acceleration filter and the energy filter. The acceleration filter requires two initial values and two final values, which are difficult to determine. The overwrapping time window technique is employed so that the boundary conditions can be ignored. The regularization factor is defined by the target frequency and the target accuracy.

The energy filter is a low-pass filter as well. This is reasonable because the displacement-strain energy relationship amplifies high frequency noise, since the strain is the first derivative of the displacement. The boundary condition is identical to that of the continuum and is always satisfied. The regularization factor is defined by the target temporal frequency and the target accuracy, which define the regularization factor of the acceleration filter. Consistency between the two filters is satisfied by the relationship between the regularization factors.

The bias compensation is served to remove the bias from the solutions. The inverse analysis using EEE has been used with the adaptive filters and the flutter derivative extraction scheme. The biases of the fields are ignorable and are easy to remove. However, the bias of the inverse analysis using EEE for the continuum is not ignorable because the noise is amplified by the inverse analyzing process. The bias compensation is realized by reconstruction of the bias terms. The

reconstructed bias terms are functions of the noise variances, the FEM model and the characteristics of the filters. When the variances of the noise sources are precisely known, the bias of the solution is perfectly eliminated. Even if the noise variance is unknown, the bias compensation can approximately reconstruct the bias terms.

The aluminum plate example demonstrates the acceleration-energy filter and the bias compensation. This example shows that the two schemes improve the solution of the inverse analysis using EEE. The window size, the measuring time, the damping of structures, the various noise levels, the target frequency and the noise of specific frequency are verified by the aluminum example.

The schemes are applied to a medical imaging application using ultrasonic equipment. Ultrasonic elastography helps to find tumors and cancers. The Helmholtz equation is used to analyze tissues of human bodies, which is the incompressible continuum. The inverse analysis and the solution stabilizers for the Helmholtz equation are derived and are found to be nearly identical to these of the general continuum. the acceleration-energy filter and the bias compensation were shown to work properly for the inverse analysis of the Helmholtz equation.

# Reference

#### [Al-Alaoui 2007]

Al-Alaoui M A, "Linear phase low-pass IIR digital differentiators", *Signal Processing*, vol. 55, no. 2, 697-706, 2007

#### [Banan 1995]

Banan M R, Hjelmstad K D, "Time-domain parameter estimation algorithm for structures II: numerical simulation studies", *Journal of Engineering Mechanics*, 121(3), pp. 435-447, 1995

#### [Bercoff 2003]

Bercoff J, Chaffai S, Tanter M, Sandrin L, Catheline S, Fink M, Gennison J L, Meunier M, "In vivo breast tumor detection using transient elastography", *Ultrasound in Med. & Biol.*, vol. 29, no. 10, pp. 1387-1396, 2003.

#### [Bercoff 2004]

Bercoff J, Tanter M, Fink M, "Supersonic shear imaging: a new technique for soft tissue elasticity mapping", *IEEE Trans. Ultrason. Ferroelectr. Freq. Control*, vol. 51, pp. 396-409, 2004.

#### [Bracewell 2000]

Bracewell R N, "The Fourier transform and its application", Third Edition, WBC/McGraw Hill, Boston, 2000

## [Bui 1994]

Bui H D, "Inverse problems in the mechanics of materials: An introduction", *CRC Press: Boca Raton*, 1994.

#### [Choi 2013]

Choi S W, Kim B R, Lee H M, Kim Y, Park H S, "A deformed shape monitoring model for building structures based on a 2D laser scanner", *Sensors*, vol. 13, pp. 6746-6758, 2013

## [Chopra 2007]

Chopra A K, "Dynamics of Structures - Theory and Application to Earthquake Engineering", Third Edition, *Pearson Education Inc. Upper Saddle River, New Jersey*, 2007

### [Ding 2011]

Ding J, Ding F, "Bias compensation-based parameter estimation for output error moving average system", *International Journal of Adaptive Control and Signal Processing*, vol. 25, pp. 1100-1111, 2011

#### [Doebling 1996]

Doebling S W, Farrar C R, "Damage identification and health monitoring of structural and mechanical systems from change in their vibration characteristics: a literature review", *Los Alamos National Laboratory Report*, LA-13070-MS, 1996

#### [Doyley 2012]

Doyley M M, "Model-based elastography: a survey of approaches to the inverse elasticity problem", *Physics in Medicine and Biology*, vol. 57, R35, 2012.

#### [Fung 1994]

Fung Y C, "A First Course in Continuum Mechanics", *PRENTICE HALL, Englewood Cliffs, Yew Jersey*, 1994

# [Ge 1998a]

Ge L, Soong T T, "Damage identification through regularization method I: theory", *Journal of Engineering Mechanics*, vol. 124, no. 1, pp. 103-108, 1998

# [Gear 2009]

Gear J M, Goodno B J, "Mechanics of Materials", Seventh Edition (SI Edition), *Cengage* Learning, *Stanford*, USA, 2009

#### [Ge 1998b]

Ge L, Soong T T, "Damage identification through regularization method II: applications", *Journal of Engineering Mechanics*, vol. 124, no. 1, pp. 109-116, 1998

#### [Goodwin 1984]

Goodwin G C, Sin K S, "Adaptive filtering prediction and control", *Prentice Hall, Inc., NewSouth Wales*, 1984

#### [Graff 1991]

Graff K F, "Wave Motion in Elastic Solids", Dover Publications, 1991.

#### [Groetsch 1984]

Groetsch C W, "The theory of Tikhonov regularization for Fredholm equations of the first kind", *Pitman Advanced Publishing: Boston*, 1984.

#### [Hamming 1989]

Hamming R W, "Digital filters", Third Edition, Prentice-Hall, Englewood Cliffs, NJ, 1989

#### [Hansen 1992a]

Hansen P C, "Analysis of discrete ill-posed problems by means of the L-curve", *SIAM review*, vol. 34, no. 4, pp. 561-580, 1992.

#### [Hansen 1992b]

Hansen P C, Sekii T, Shibahashi H, "The modified truncated SVD method for regularization in general form", *SIAM J. Sci. Stat. Comput.*, vol. 13, pp. 1142-1150, 1992

#### [Hansen 1997]

Hjelmstad K D, Shin S, "Damage detection and assessment of structures from static response", *Journal of Engineering Mechanics*, 123(6), pp. 568-576, 1997

# [Hansen 1998]

Hansen P C, "Rank-deficient and discrete ill-posed problems: numerical aspects of linear inversion", *SIAM*, *Philadelphia*, 1998

#### [Haykin 2002]

Haykin S, "Adaptive filter theory", Forth Edition, Pearson Education, Inc., 2002

# [Hjelmstad 1995]

Hjelmstad K D, Banan M R, and Banan M R, "Time-domain parameter estimation algorithm for structures. I: computational aspects", *Journal of Engineering Mechanics*, vol. 121, no. 3, pp. 424-434, 1995.

#### [Hjelmstad 1996]

Hjelmstad K D, "On the uniqueness of modal parameter estimation", *Journal of Sound* and Vibration, vol. 192, no 2, pp. 581-598, 1996.

## [Ho 1966]

Ho B L, Kalman R E, "Effective construction of linear state-variable models from input/output functions", *Regelungstechnik*, vol. 14, no. 12, pp. 545-548, 1966

#### [Hong 2010]

Hong Y H, Kim H K, Lee H S, "Reconstruction of dynamic displacement and velocity from measured accelerations using the variational statement of an inverse problem", *Journal of Sound and Vibration*, 329 (2010) pp. 4980-5003, 2010.

#### [Hong 2012]

Hong Y H, "Reconstruction of Dynamic Displacement and Velocity based on the Variation Statement of an Inverse Problem from Measured Acceleration with Special Application to the Extraction of Flutter Derivatives", *Ph. D thesis, Seoul National University, Korea*, 2012.

# [Hong 2013]

Hong Y H, Lee S G, Lee H S, "Design of the FEM-FIR filter for displacement reconstruction using accelerations and displacements measured at different sampling rates", *Mechanial Systems and Signal Processing*, Vol. 38, No. 2, pp. 460-481, 2013.7.

# [Hughes 1987]

Hughes T J R, "the finite element method: linear static and dynamic finite element analysis", *Prentice-Hall, Englewood Cliff, NJ*, 1987

# [Ikenoue 2009]

Ikenoue M, Kanae S, Yang Z J, Wada K, "Identification of errors-in-variables model via bias-compensated instrumental variables type method", *International Journal of Onnovative Computing, Information and Control*, vol. 5, pp. 29-40, 2009

#### [Johnson 1987]

Johnson C, "Numerical solution of partial differential equations by the finite element methods", *Cambridge University Press: New York*, 1987.

# [Kang 2005]

Kang J S, Park S K, Shin S, Lee H S, "Structural System Identification in Time Domain using Measured Acceleration", *Journal of Sound and Vibration*, vol. 288, pp. 215-234, 2005.11.

#### [Kim 2011]

Kim J, "System identification of civil engineering structures though wireless structureal monitoring and subspace system identification methods", *Ph. D Thesis, University of Michigan*, 2011

#### [Kreyszig 2006]

Kreyszig E, "Advanced Engineering Mathematics", Ninth Edition, John Wiley & Sons, Inc., New Jersey, 2006

#### [Kumar 1996]

Kumar B, Choudhury D R, Kumar A, "On the design of linear phase, FIR integrators for midband frequencoes", *IEEE, Transacation of signal processing*, vol. 44(10), pp. 2391-2395, 1996

#### [Lawler 2006]

Lawler G F, "Introduction to Stochastic Processes", Second Edition, Chapman & Hall/CRC, New York, 2006

# [Lee 1999]

Lee H S, Kim Y H, Park C J, Park H W, "A new spatial regularization scheme for the identification of geometric shapes of inclusions in finite bodies", *International Journal for Numerical Methods in Engineering*, vol. 46, no. 7, pp. 973-992, 1999.

## [Lee 2000]

Lee H S, Park C J, Park H W, "Identification of geometric shapes and material properties of inclusions in two-dimensional finite bodies by boundary parameterization", *Computer Methods in Applied Mechanics and Engineering*, vol. 181, no. 1-2, pp. 1-20, 2000.1.

#### [Lee 2010]

Lee H S, Hong Y H, Park H W, "Design of an FIR Filter for the Displacement Reconstruction Using Measured Acceleration in Low-frequency Dominant Structures", *International Journal for Numerical Methods in Engineering*, vol. 82, pp. 403-434, 2010.

#### [Lubinski 1996]

Lubinski M A, Emelianov S Y, Raghavan K R, Yagle A E, Skovoroda A R, O'Donnell M, "Lateral displacement estimation using tissue incompressibility", *IEEE Trans. Ultrason. Ferroelectr. Freq. Control*, vol. 43, pp. 247-255, 1996.

# [Luengberger 1979]

Luengberger D G, "Introduction to Dynamic Systems: Theory, Models & Applications", John Wiley & Son, Inc., New York, 1979

# [Mashreghi 2014]

Mashreghi A, Yazdi H S, "A recursive algorithm for optimizing differentiation", *Journal of Computational and Applied Mathematics*, vol. 263, pp. 1-13, 2014

#### [McGuiare 2000]

McGuire W, Gallagher R H, Ziemian R D, "Matrix Structural Analysis", Second Edition, *John Wiley & Sons, Inc., New York*, 2000

# [Nguyen 1993]

Nguyen H L, Sibille P, Garnier H, "A new bias compensating least-squares method for identification of stochastic linear system in presence of colored noise", *Proceedings of 32nd conference of decision and control, San Antonio, Texas*, 1993

# [Ophir 1991]

Ophir J, Cespedes I, Ponnekanti H, Yazdi Y, Li X, "Elastography: a quantitative method for imaging the elasticity of biological tissues", *Ultrason. Imaging*, vol. 13, pp. 111-134, 1991.

#### [Oppenheim 1999]

Oppenheim A V, Schafer R W, Buck J R, "Discrete-time signal processing", Second Edition, *Prentice-Hall, upper Saddle River, NJ*, 1999

#### [Park 2001]

Park H W, Shin S B, Lee H S, "Determination of an optimal regularization factor in system identification with Tikhonov function for linear elastic continua", *International Journal for Numerical Methods in Engineering*, vol. 51, no. 10, pp. 1211-1230, 2001.

# [Park 2002]

Park H W, "Regularization techniques in system identification for damage assessment of structures", *PhD Thesis, Seoul National University, Seoul Korea*, 2002.

## [Park 2006a]

Park Y C, "A stabilizating sheme for the dynamic analysis of slack cables", *Master Thesis, Seoul National University*, 2006

# [Park 2006b]

Park E Y, Maniatty A M, "Shear modulus reconstruction in dynamic elastography: time harmonic case", *Phys. Med. Biol.*, vol. 51, pp. 3697-3721, 2006.

# [Park 2007]

Park H W, Park M W, Ahn B K, Lee H S, "1-norm based regularization scheme for system identification of structures with discontinuous system parameters", *International Journal for Numerical Methods in Engineering*, vol. 69, no. 3, pp. 504-523, 2007.

#### [Park 2008]

Park S K, Park H W, Shin S, Lee H S, "Detection of Abrupt Structural Damage Induced by an Earthquake Using a Moving Time-Window Technique", *Computers and Structures*, vol. 86, no. 11-12, pp. 1253-1265, 2008.

#### [Park 2009]

Park E Y, Maniatty A M, "Finite element formulation for shear modulus reconstruction in transient elastography", *Inverse Problems in Science and Engineering*, vol. 17, pp. 605-626, 2009.

#### [Park 2012]

Park S K, "Time domain SI scheme for time-invariant and time-variant structural system", *Ph. D thesis, Seoul National University, Korea*, 2012.

## [Park 2013]

Park H S, Kim J M, Choi S W, Kim Y, "A wireless laser displacement sensor node for structural health monitoring", *Sensores*, vol. 13, pp. 13204-13216, 2013

#### [Parker 1990]

Parker K J, Huang S R, Musulin R A, Lerner R M, "Tissue-response tomechanical vibrations for sonoelasticity imaging", *Ultrasound Med. Biol.*, vol. 16, pp. 241-246, 1990.

# [Ponnekanti 1994]

Ponnekanti H, Ophir J, Cespedes I, "Ultrasonic-imaging of the stress-distribution in elastic media due to an external compressor", *Ultrasound Med. Biol.*, vol. 20, pp. 27-33, 1994.

# [Qi 2014]

Qi X, Lichti D, El-Badry M, Chow J, Ang K, "Vertical dynamic deflection measurement in concrete beams with Microsoft Kinect", *Sensors*, vol. 14, pp. 3293-3307, 2014

#### [Rabiner 1975]

Rabiner L R, Gold B, "Theory and application of digital signal processing", *Prentice-Hall, Englewood Cliffs, NJ*, 1975

#### [Raghavan 1994]

Raghavan K R, Yagle A E, "Forward and inverse problems in elasticity imaging of soft-tissues", *IEEE Trans. Nucl. Sci.*, vol. 41, pp. 1639-1648, 1994.

#### [Rao 1996]

Rao S S, "Engineering Optimization: Theory and Practice", Third Edition, *John Wiley* & *Son, Inc., New York*, 1996

#### [Reddy 1990]

Reddy M R R, Kumar B, Dutta Roy S C, "Design of efficient second and higher order FIR digital differentiators for low frequencies", *Signal Processing*, vol. 20, pp. 219-225, 1990

# [Richards 2009]

Richards M S, Barbone P E, Oberai A A, "Quantitative three-dimensional elasticity imaging from quasi-static deformation: a phantom study", *Phys. Med. Biol.*, vol. 54, pp. 757-779, 2009.

#### [Sarvazyan 1998]

Sarvazyan A, Rudenko O, Swanson S, "Shear wave elasticity imaging: a new ultrasonic technology of medical diagnostics", *Ultrasound Med. Biol.*, vol. 24, pp. 1419-1435, 1998.

#### [Skovoroda 1995]

Skovoroda A R,Aglyamov S R, "Reconstruction of elastic properties of soft biological tissues exposed to low-frequencies disruption", *Biofizika*, vol. 40, pp. 1329-1334, 1995.

# [Sohn 2004]

Sohn H, Farrar C R, Hemez F M, Shunk D D, Stinemates S W, Nadler B R, Czarnecki J J, "A review of structural health monitoring literature: 1996-2001", *Los Alamos National Laboratory report*, LA-13976-MS, 2004Vertroni F, Capecchi D, "Damage detection in beam structures based on frequency measurements", *Journal of Engineering Mechanics*, 126(7), pp. 761-768, 2000

## [Timoshenco 1959]

Timoshenco S P, Woinowsky-Krieger S, "Theory of Plates and Shell", Second Edition, *McGraw-Hill, New York*, 1959

#### [Timoshenco 1970]

Timoshenco S P, Goodier J N, "Theory of Elasticity", Third Edition, McGraw-Hill, New York, 1970

#### [Vertroni 2000]

Vertroni F, Capecchi D, "Damage detection in beam structures based on frequency measurements", *Journal of Engineering Mechanics*, 126(7), pp. 761-768, 2000

# [Wahbeh 2003]

Wahbeh A M, Carffrey J P Masri S F, "A vision-based approach for the direct measurement of displacements in vibrating system", *Smart Materials and Structures*, vol. 12, pp. 758-794, 2003

# [Yeo 2000]

Yeo I H, Shin S, Lee H S, Chang S P, "Statistical damage assessment of framed structures from static responses", *Journal of Engineering Mechanics*, ASCE, vol. 126, no. 4, pp. 414-421, 2000.

#### [Yoshitomi 2009]

Yoshitomi S, Takewaki I, "Noise-bias compensation in physical-parameter system identification under microtremor input", *Engineering Structures*, vol. 31, pp. 580-590, 2009

# [Zhang 2011]

Zhang Y, Cui G, "Bias compensation methods for stochastic system with colored noise", *Applied Mathematical Modeling*, vol. 35, pp. 1709-1716, 2011

# 동적 변위를 사용하는 역해석 문제에서 Equation Error Estimator 안정화를 위한 가속도-에너지 필터 및 편향성 보정 기법

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박 광 연

본 학위 논문은 동적 변위를 이용해 선형탄성 연속체를 역해석하는 방법 중 Equation Error Estimator (EEE)를 이용한 기법을 안정화 시키기 위한 이 론을 제시한다. 여기서 역해석은 미지의 강성을 추정하는 기법을 말한 다. 안정화 기법은 가속도-에너지 필터와 편향성보정으로 이루어져 있다. 가속도-에너지 필터는 측정 변위가 가져야 할 물리적 조건을 이용해 역해석에서 발생하는 해의 부적합성(III-posedness)을 안정화 한다. 여기 서 사용한 물리적 조건은 가속도 유한 조건과 변형 에너지 유한 조건이 다. 이 기법들은 해의 부적합성을 안정화 시키기 위해 기존에 널리 쓰 이고 있는 특이치분해 절단법 (truncated singular value decomposition; TSVD),

L<sub>1</sub>-norm 정규화, L<sub>2</sub>-norm 정규화 (Tilhonov 정규화) 등의 단점을 보완하기 위해 개발 되었다. 정규화 기법들은 연속체 내부에 강성이 작은 부분이 포함 된 경우 이를 효과적으로 탐지할 수 있다. 하지만 연속체 내부에 강성이 큰 부분이 포함 된 경우 정상적으로 작동하지 않는 것을 확인 하 였다. 인체 조직의 암이나 차량 서스펜션의 경화현상, 콘크리트 내부에 이물일지 포함 된 경우 등과 같은 것들이 이에 해당한다. 이에 반해 가 속도-에너지 필터는 연속체 내부의 강성 분포에 상관없이 언제나 정상적 으로 작동한다. 가속도-에너지 필터는 가속도 필터와 에너지 필터로 분 리할 수 있다.

가속도 필터는 측정 변위의 2계 미분이 부분연속 함수공간 안에 있 다는 정보를 정규조건으로 만들어진 필터다. 2계 미분한 변위, 즉 가속 도는 실제로 부분연속 함수가 아니지만 충격하중이 없다고 가정하면 부 분연속 함수로 볼 수 있다. FDM-FIR (Finite impulse response using finite difference method) 을 이용해 이산화된 가속도 필터는 가속도의 초기값과 최종값을 알아야 풀 수 있는 문제이다. 하지만 대부분의 경우 가속도의 초기값과 최종값은 알기 어려우며 이를 해결하기 위해 이동 시간창 기법 을 도입해 경계조건이 필요 없는 문제로 바꾼다. FDM-FIR로 이산화된 가속도 필터는 고주파 잡음을 걸러내는 일반적인 잡음제거 필터의 형태 를 갖는다. 하지만 가속도 필터는 에너지 필터와 일관성(consistency)을

유지할 수 있도록 하는 물리적 의미를 가지고 있다. 이는 다른 디지털 필터가 갖지 못한 장점이다.

에너지 필터는 연속체의 내부변형 에너지가 유한해야 한다는 조건을 정규조건으로 만들어진 필터다. FEM을 이용해 이산화 된 에너지 필터 는 영상처리분야에서 많이 사용하는 공간 디지털 필터와 흡사한 형태를 갖지만 몇 가지 차별성이 있다. 첫째는 필터의 경계조건이 연속체 방정 식의 경계조건과 같다는 점이다. 이로 인해 에너지 필터의 경계조건은 연속체가 가질 수 있는 모든 종류의 경계조건에 대해 유효하다. 둘째는 FEM 요소의 형태적 측면이 있다. 공간 디지털 필터는 측정점이 사각형 형태로 분포하지 않으면 사용하기 어렵다. 하지만 에너지 필터는 FEM 모형이 가지고 있는 절점 연결 정보를 함께 포함하기 때문에 복잡한 측 정점 분포에도 적용 가능하다. 셋째는 에너지 필터가 가진 물리적 의미 를 통해 가속도 정규화와의 일관성을 유지할 수 있는 점이다.

가속도 필터와 에너지 필터는 연속체의 시-공간 주파수의 관계에 의 한 일관성을 유지하도록 설계 되어야 한다. 시-공간 주파수의 관계를 무 시하고 만들어진 필터는 역해석의 정확도를 감소 시킨다. 가속도 필터 와 에너지 필터가 가지고 있는 물리적 의미는 일관성을 유지 할 수 있는 매개체 역할을 하며 이는 다른 잡음제거 필터가 갖지 못한 장점이다.

EEE를 이용한 역해석 기법은 해의 불안정성과 동시에 해의 편향성

(bias of solution) 문제를 가지고 있다. 기존 연구들은 대부분 편향성을 무 시하였는데 이는 무시할 수 있을 만큼 작은 경우가 많기 때문이다. 하 지만 선형 탄성 연속체의 역해석을 위한 EEE에 발생하는 해의 편향성은 무시할 수 없을 정도로 크다. 편향성 보정 기법은 이러한 해의 편향성 을 제거 하는 기법으로 측정잡음의 분산을 알면 편향성을 완벽하게 제거 할 수 있다. 잡음의 분산을 모르는 경우에는 잡음 분산을 추정하여 편 향성을 제거 할 수도 있다.

기법의 효과를 확인하기 위해 알루미늄 판 예제와 초음파 영상의학 예제를 도입한다. 이 예제를 통해 가속도-에너지 필터와 편향성 보정 기 법을 실제 적용하는 방법을 제시하고 그 유효성을 증명한다.

주요어: Equation Error Estimation (EEE), 시스템 확인 기법, 역해석, 물성치 추정, 가속도-에너지 필터, 시공간 필터, 편향성보정기법, 선형탄성 연속체, 영상의학

학번: 2010-30239