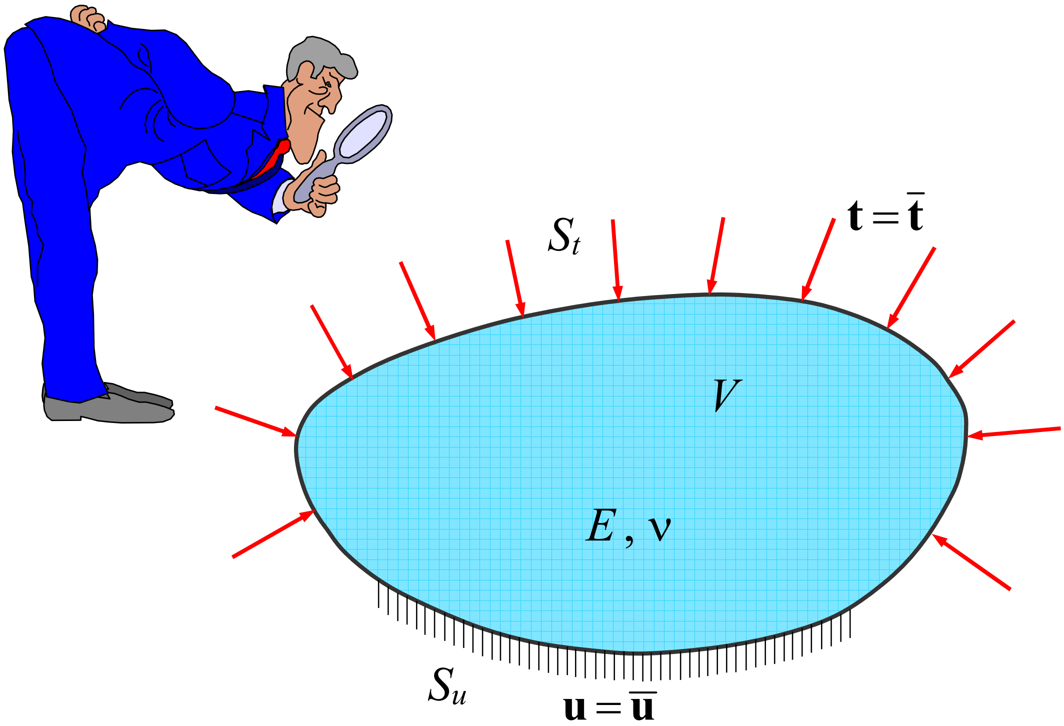
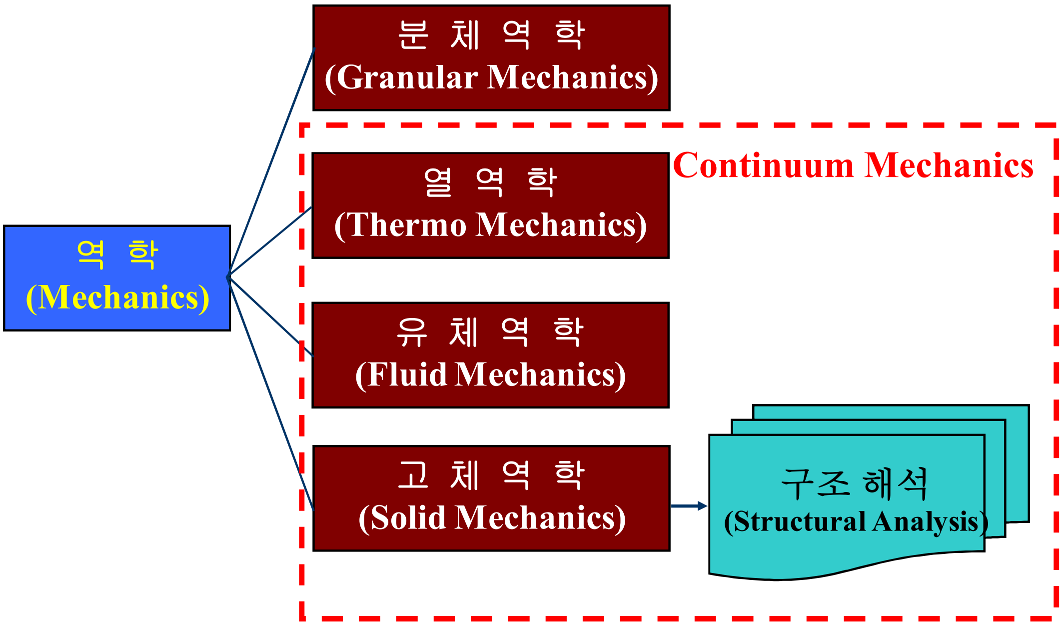
Chapter 1
Fundamentals in Elasticity



1.0 **Introduction**

- Classification of Classic Mechanics



- Definition of Mechanics by Encyclopedia Britannica, 2004

  Science concerned with the motion of bodies under the action of forces, including the special case in which a body remains at rest. Of first concern in the problem of motion are the forces that bodies exert on one another. This leads to the study of such topics as gravitation, electricity, and magnetism, according to the nature of the forces involved. Given the forces, one can seek the manner in which bodies move under the action of forces; this is the subject matter of mechanics proper.

- What is Classic Mechanics?

  The motion of bodies follows the Newton's laws of motion.

  - The law of inertia
  - The law of acceleration : $F = ma$
  - The law of action and reaction

- Who started?
  - Galilei Galileo (1564 ~ 1642)

    He tried to explain motions of bodies based on observation or experimentation. His formulation of (circular) inertia, the law of falling bodies, and parabolic trajectories marked the beginning of a fundamental change in the study of motion. He insisted that the rules of nature should be written in the language of mathematics, which changed natural philosophy from a verbal, qualitative account to a mathematical one He utilized experimentation as a recognized method for discovering the facts of nature. Unfortunately, he did not have mathematical tools known as "calculus" that deals with differentiation and integration, and thus he failed to describe motions of bodies in a complete mathematical way.

  - Issac Newton (1642~ 1727)

    He completed the mathematical formulations of the classic mechanics that Galileo tried by the virtue of differentiation and integration, which also independently proposed by a German mathematician Gottfried Wilhelm Leibniz (1646~1716) almost at the same time..

  - Who is next?

    We can name hundreds of great scientists such as

    - ✓ R. Hooke (1635~1703): Hooke's Law
    - ✓ Jakob. Bernoulli (1655~1705): beam theory, study on catenary
    - ✓ Daniel Bernoulli (1700~1782): Bernoulli's principle for the inviscid flow
    - ✓ L. Euler (1707~1783): Euler equation, Euler buckling load…
    - ✓ T. Young (1773~1829): Young's modulus, interference of light…
    - ✓ A. Cauchy(1789~1857): Cauchy's strain, functions of a complex variable…

- Fluids and Solids
  - Difference in Material Properties
    - ✓ Fluid flows because $G << E$.
    - ✓ Solid does not because $G \approx E$.
    - ✓ Inviscid fluid: $G \approx 0$
    - ✓ Viscous fluid: $0 << G << E$

- Fundamental principles in the fluid and solid mechanics are identical except material properties.
- The fluid and solid mechanics can be formulated from exactly the same framework.
- The textbook by Malvern (Introduction to the mechanics of a continuous media) deals with the solid, fluid and thermo mechanics simultaneously.

● Elasticity, Plasticity and Linear Problems

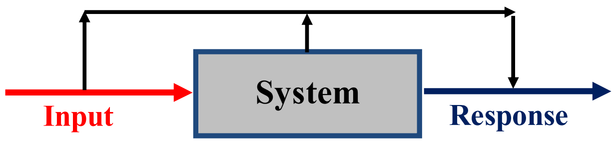
- Bodies with Elastic Material: Recovers the original shapes of a body when external loads are removed.
- Bodies with Plastic Material: Cannot recover the original shapes of a body when external loads are removed. Permanent deformation remains in bodies.
- Linear Problems: The principle of the superposition is valid.

$$A \rightarrow \alpha,\ B \rightarrow \beta \ \text{ then } \ A + B \rightarrow \alpha + \beta$$

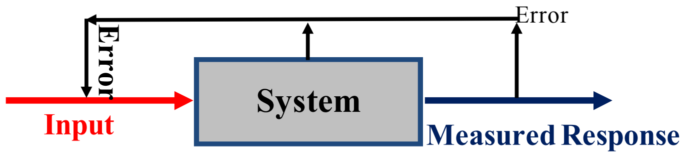
- The elasticity problems may be either linear or nonlinear.

● Classification of Engineering Problems

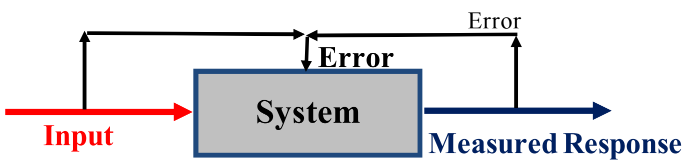
- **Direct Problems : $\nabla \cdot (k \nabla \cdot \boldsymbol{u}) = f$ (Analysis)**



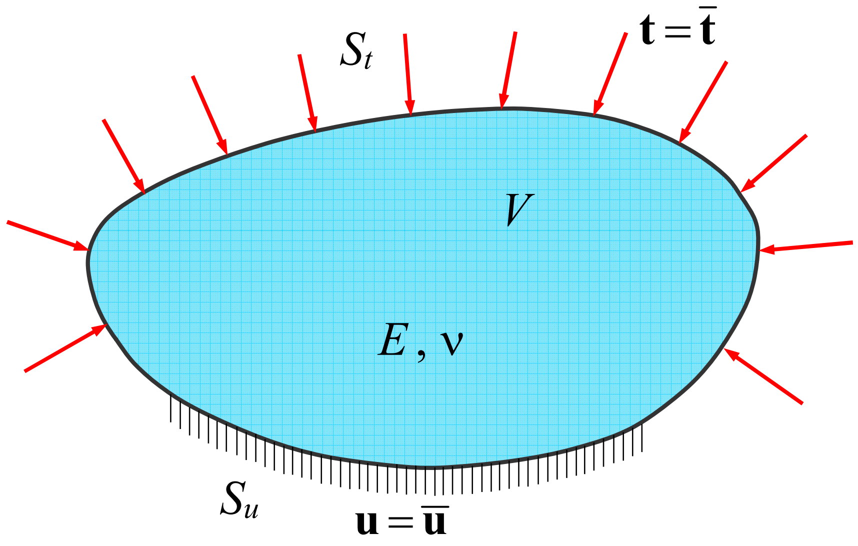
- **Inverse Problems : Reconstruction**



- **Inverse Problems : System Identification**



1.1 Problem Definition



- Prescribed Values
  - Domain V and Boundary S
  - Material properties
  - Boundary Conditions : $\mathbf{u}=\bar{\mathbf{u}}$ on S_u , $\mathbf{t}=\bar{\mathbf{t}}$ on S_t, where $S_u \cup S_t = S$ and $S_u \cap S_t = \varnothing$

- Unknowns in domain:
  - Stress ($\boldsymbol{\sigma}$)
  - Strain ($\boldsymbol{\varepsilon}$)
  - Displacement ($\mathbf{u}$)

- We have 15 unknowns, and thus have to derive 15 equations to solve the elasticity problems. Since we determine the 15 unknowns in the domain from the prescribed boundary conditions, the elasticity problems are a type of mixed boundary value problems.

- What we have to study during this class:
  - Stress ($\boldsymbol{\sigma}$): force per unit area developed in the body by the action of external forces
  - Strain ($\boldsymbol{\varepsilon}$): Deformation of body
  - Displacement ($\mathbf{u}$): Changes in positions of the body caused by the motion of the body
  - Relationships among the unknowns such as equilibrium, strain-displacement relationship, strain-stress relationship

- Equilibrium equation:

  Mainly concerns the equilibrium state of a given body under the actions of external forces, and expressed in terms of stress.

$$\int_S \mathbf{t} dS = 0$$

- Strain-displacement relation:

  Defines deformation of a given body, and relate strain to displacement.

$$\varepsilon = f(u)$$

- Stress-strain relation:

  Represent the material properties of a given body such as the Hooke's Law.

$$\sigma = g(\varepsilon)$$

- **Types of Engineering Problems**
  - Boundary Value Problem (BVP): The unknowns are determined from the prescribed values on the boundary of a given domain. Especially when the boundary conditions are expressed in terms of the unknowns themselves as well as their derivative, then the problems are referred to the mixed BVP is given as:

$$\frac{d^2\phi(x)}{dx^2} + f = 0 \text{ for } 0 < x < l \text{ and } \phi(0) = 0 \ \frac{d\phi(l)}{dx} = 0$$

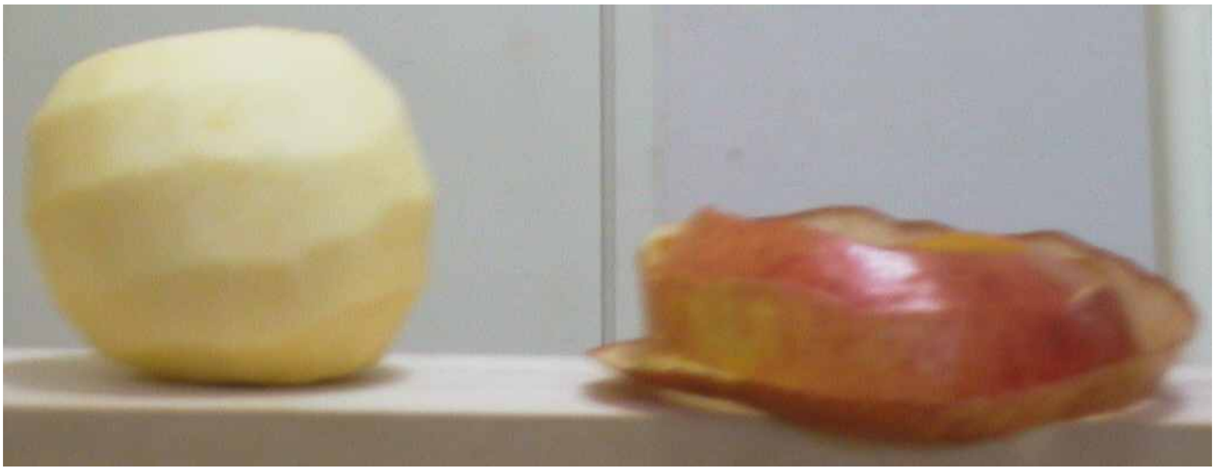
  - Initial value problems (IVP): The unknowns of a given problem are determined from initial values prescribed at a reference time, usually at $t = 0$.

$$\frac{d^2\phi(t)}{dt^2} - f(t) = 0 \text{ for } 0 < t \text{ and } \phi(0) = 0 \ \frac{d\phi(0)}{dt} = 0$$

  - Initial-boundary Value Problems: we have to utilize both the initial and boundary values to solve a given problem.

$$\frac{d^2\phi(t,x)}{dt^2} = \frac{d^2\phi(t,x)}{dx^2} + f(t,x) \text{ for } 0 < t \text{ and } \phi(0,x) = 0 \ \frac{d\phi(0,x)}{dt} = 0 \text{ and } 0 < x < l \text{ and } \phi(t,0) = \phi(t,l) = 0$$

1.2. Domain and Boundary



1.3. Definition of Continuum

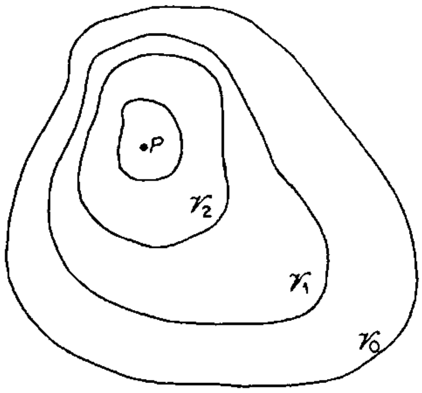
- Definition of a continuous set

  A set A is called as a continuous set if there always exists $c \in A$ which lies between a and b for all $a, b \in A$.

  Example: real numbers

- Continuous distribution

  A physical quantity ρ is said to be continuously distributed if the following limit is uniquely defined everywhere in the given domain V.



$$\rho(P) = \lim_{n \to \infty, V_n \to 0} \frac{M_n}{V_n}$$

  where M_n is the sum of the quantity in V_n, and $V_n \subset V_{n-1}$, $P \in V_n$, $V_0 \equiv V$.

- Continuous body or continuum in a mathematical sense

  A body is called as continuum if material particles are continuously distributed in the body or there exists the continuous density function.

- Continuous body or continuum in a real sense

  Instead of $V_n \to 0$ as $n \to \infty$, V_n approaches a finite number ω as $n \to \infty$. Then,

$$\lim_{n \to \infty, V_n \to \omega} \left| \rho(P) - \frac{M_n}{V_n} \right| < \varepsilon$$

  where ε is an acceptable variability. You may consider ω as the smallest volume you can differentiate with your own naked eyes. A body is referred to as a continuum if material particles are distributed in a continuous fashion. Therefore, conceptually, you can pick a material particle between any two material particles in the continuous body.

1.4. Fundamental Laws

- Newton's Laws of Motion
  - The momentum of an object (m**v**) is constant unless an outside force acts on the object; this means that any object either remains at rest or continues uniform motion in a straight line unless acted on by a force.
  - The time rate of change of the momentum of an object is equal to the force acting on the object.
  - For every action (force) there is an equal and opposite reaction (force).

- The 1st law of thermodynamics : The conservation of energy

- The 2nd law of thermodynamics

  Defines the direction of energy flow. That is, energy always flows from the high level to the low level spontaneously. Work should be applied to reverse the spontaneous energy flow.

1.5. Axioms

1. Newton's laws of motion and the 1st and 2nd law of thermodynamics are valid.

2. A material continuum remains a continuum under the action of force.

3. Stress and strain can be defined everywhere in the body.

4. The stress at a point is related to the strain and the rate of change of strain at the same point.

$$\boldsymbol{\sigma}(P) = \mathbf{f}(\boldsymbol{\varepsilon}(P), \dot{\boldsymbol{\varepsilon}}(P)) \quad \forall P \in V$$

For a rate-independent material, $\boldsymbol{\sigma}(P) = \mathbf{f}(\boldsymbol{\varepsilon}(P)) \quad \forall P \in V$ where the temporal rate $= \dfrac{d()}{dt}$ and spatial change $= \dfrac{\partial()}{\partial x}, \dfrac{\partial()}{\partial y}, \dfrac{\partial()}{\partial z}$.

- The consequence of the last axiom?

  If the stress at a point **were** influenced by strains at the other points, the stress-strain relation should include the spatial derivatives of the strain to represent the spatial change of strain, which results in differential equations. Since, however the last axiom states that the stress at a point depends only the strain at the same point, the stress at a point is independent of the spatial derivatives of strain. The stress-strain relation representing material properties of a given domain is expressed algebraically in the spatial domain, and 6 equations out of 15 governing equations become algebraic equations rather than differential equations, which make the elasticity problems much simpler. Since the rate-dependent material indicates that the stress at a point depends on temporal rate change of strain at the point, the stress-strain relation becomes differential equations in the time domain.

1.6. Tensors and Operations

- Tensorial notation : **A**

- Indicial notation : A_i , A_{ij} , A_{ijk} , A_{ijkl}

- Summation notation : the repeated subscripts or superscripts in a term denotes summation.

  The repeated index is called as "dummy index".

$$\sum_{i=1}^{n} A_i B_i = A_1 B_1 + A_2 B_2 + \cdots + A_n B_n = A_i B_i = A_k B_k$$

$$A_i A_i = A_1^2 + A_2^2 + \cdots + A_n^2 \neq A_i^2$$

$$A_{ij} B_j = \sum_{j=1}^{n} A_{ij} B_j = A_{i1} B_1 + A_{i2} B_2 + \cdots + A_{in} B_n$$

$$A_{ij} B_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} B_{ij} = \sum_{i=1}^{n} (A_{i1} B_{i1} + A_{i2} B_{i2} + \cdots + A_{in} B_{in})$$

- Dot product

  A dot product of any two variables represents the summation on the last index of the first variables and the first index of the second variable.

$$\mathbf{A} \cdot \mathbf{B} = A_i B_i = A_1 B_1 + A_2 B_2 + \cdots + A_n B_n$$

$$\mathbf{A} \cdot \mathbf{B} = A_{ji} B_i = A_{j1} B_1 + A_{j2} B_2 + \cdots + A_{jn} B_n$$

$$\mathbf{B} \cdot \mathbf{A} = B_i A_{ij} = B_1 A_{1j} + B_2 A_{2j} + \cdots + B_n A_{nj}$$

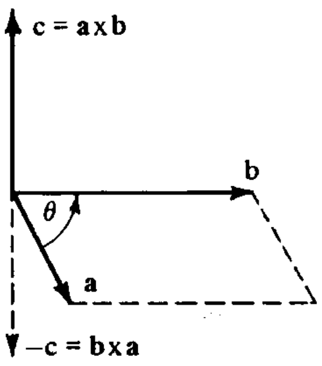
$$\mathbf{A}^T \cdot \mathbf{B} = A_{ij} B_i = A_{1j} B_1 + A_{2j} B_2 + \cdots + A_{nj} B_n$$

$$\mathbf{A} \cdot \mathbf{B} = A_{ik} B_{kj} = A_{i1} B_{1j} + A_{i2} B_{2j} + \cdots + A_{in} B_{nj}$$

- Cross product of two vectors

$$\mathbf{C} = \mathbf{A} \times \mathbf{B}$$

$$\mathbf{C} = \begin{vmatrix} \mathbf{i}_1 & \mathbf{i}_2 & \mathbf{i}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}, \quad \|\mathbf{C}\| = \|\mathbf{A}\| \|\mathbf{B}\| \sin\theta$$



- Kronecker delta

$$\delta_{ij} = \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{for } i = j \end{cases}$$

- Permutation tensor

$$e_{ijk} = \begin{cases} 1 & \text{for even permutation} \\ -1 & \text{for odd permutation} \\ 0 & \text{Any two indices are equal} \end{cases}$$

- Determinant of a matrix A

$$\mathrm{Det}(\mathbf{A}) = e_{ijk} A_{i1} A_{j2} A_{k3} = e_{ijk} A_{1i} A_{2j} A_{3k} = \mathrm{Det}(\mathbf{A}^T)$$

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i}_1 & \mathbf{i}_2 & \mathbf{i}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} = \begin{vmatrix} \mathbf{i}_1 & A_1 & B_1 \\ \mathbf{i}_2 & A_2 & B_2 \\ \mathbf{i}_3 & A_3 & B_3 \end{vmatrix} = e_{ijk} \mathbf{i}_i A_j B_k = e_{ijk} A_j B_k \mathbf{i}_i \rightarrow C_i = e_{ijk} A_j B_k$$

In general, $\mathrm{Det}(\mathbf{A}) = e_{ijk\ldots p} A_{i1} A_{j2} A_{k3} \cdots A_{pn}$

- Tensor fields

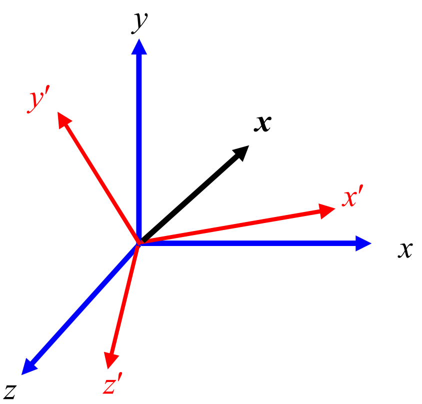
In case the coordinate rotation is defined by $x'_j = \beta_{ji} x_i$

Scalar field (tensor field of rank 0) : $\phi'(x', y', z') = \phi(x, y, z)$

Vector field (tensor field of rank 1) : $\phi'_j(x', y', z') = \beta_{ji} \phi_i(x, y, z)$

Tensor field of rank 2 : $\phi'_{ij}(x', y', z') = \beta_{im} \phi_{mn}(x, y, z) \beta_{jn}$

Tensor field of rank 4 : $\phi'_{ijkl}(x', y', z') = \beta_{ip} \beta_{jq} \phi_{pqrs}(x, y, z) \beta_{kr} \beta_{ls}$



- Gradient operator

$$\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$$

$$\nabla\phi = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})\phi = (\frac{\partial\phi}{\partial x_1}, \frac{\partial\phi}{\partial x_2}, \frac{\partial\phi}{\partial x_3}) = \frac{\partial\phi}{\partial x_i}$$

$$\nabla\cdot\nabla\phi = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})\cdot(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})\phi = (\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2} + \frac{\partial^2}{\partial x_3})\phi = \frac{\partial^2\phi}{\partial x_1^2} + \frac{\partial^2\phi}{\partial x_2^2} + \frac{\partial^2\phi}{\partial x_3^2}$$
$$= \frac{\partial^2\phi}{\partial x_i\,\partial x_i} = \nabla^2\phi$$

$$\nabla\cdot\nabla\phi = \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i}\phi = \frac{\partial^2\phi}{\partial x_i\partial x_i}$$

$$\nabla\cdot\boldsymbol{\sigma} = \frac{\partial}{\partial x_i}\sigma_{ij} = \frac{\partial\sigma_{ij}}{\partial x_i} = \frac{\partial\sigma_{1j}}{\partial x_1} + \frac{\partial\sigma_{2j}}{\partial x_2} + \frac{\partial\sigma_{3j}}{\partial x_3}$$

- Divergence (Gauss) Theorem

$$\text{div}\,\mathbf{F} = \nabla\cdot\mathbf{F} = \frac{\partial F_i}{\partial x_i} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3}$$

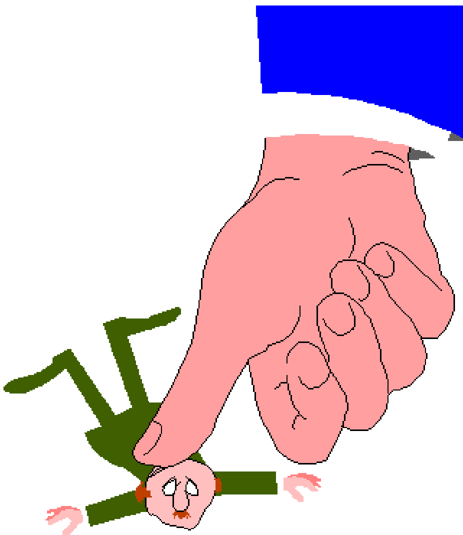
$$\int_V \nabla\cdot\mathbf{F}dV = \int_S \mathbf{F}\cdot\mathbf{n}dS$$

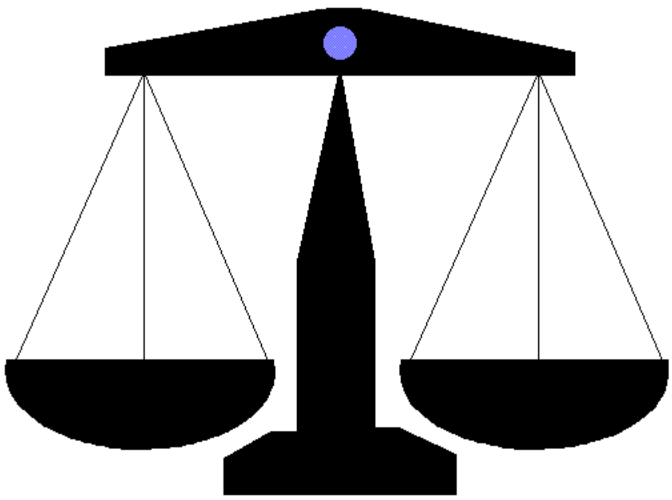
$$\int_V \nabla\cdot(g\mathbf{F})dV = \int_V \nabla g\cdot\mathbf{F}dV + \int_V g\nabla\cdot\mathbf{F}dV = \int_S g\mathbf{F}\cdot\mathbf{n}dS \rightarrow$$

$$\int_V g\nabla\cdot\mathbf{F}dV = \int_S g\mathbf{F}\cdot\mathbf{n}dS - \int_V \nabla g\cdot\mathbf{F}dV$$

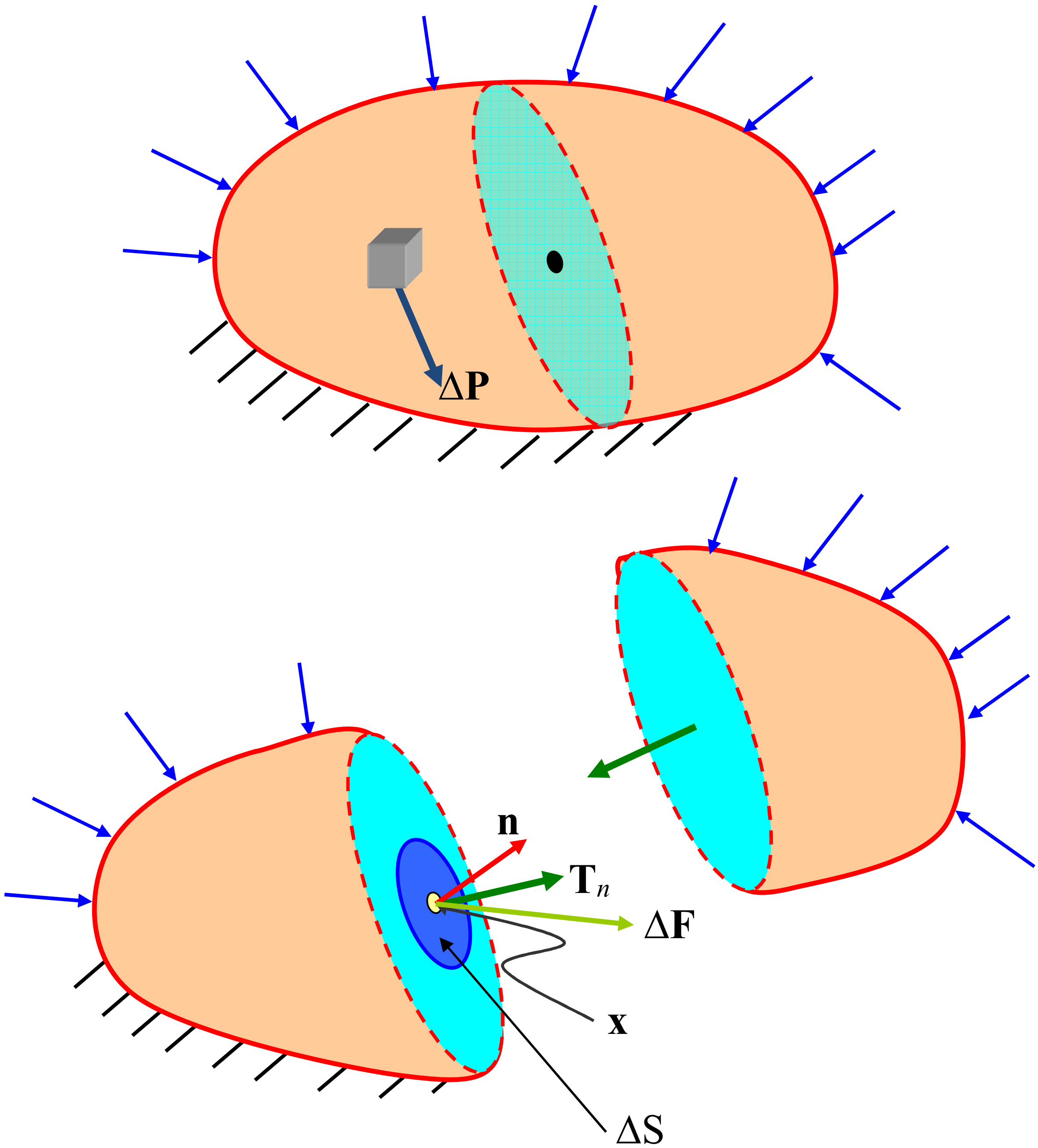
Chapter 2

Traction and Stress



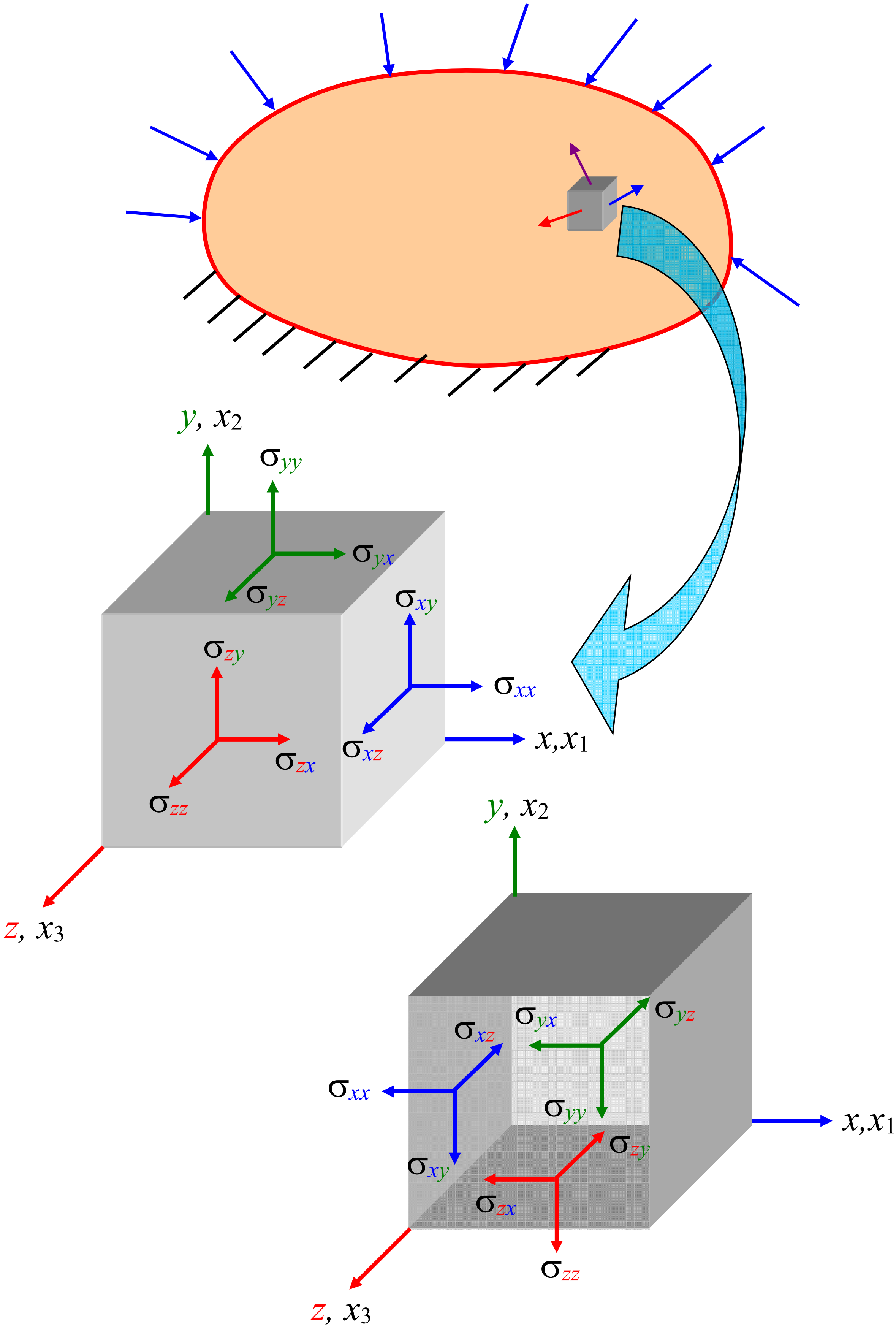


2.1. Traction and Stress

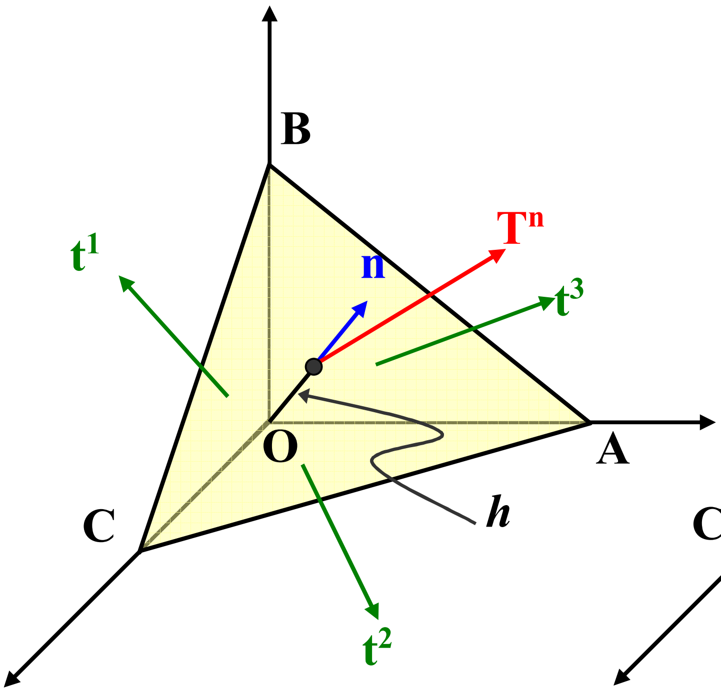


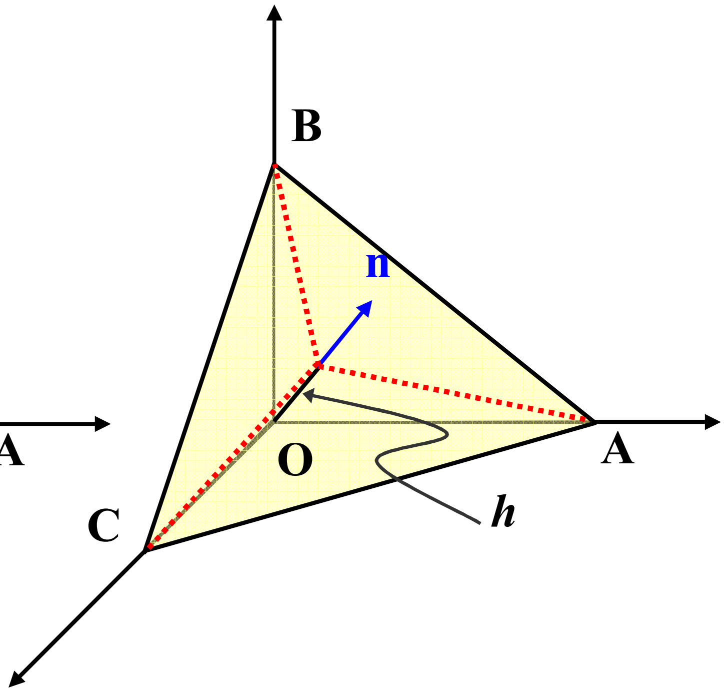
- Body Force: $\mathbf{b} = \lim_{\Delta V \to 0} \frac{\Delta \mathbf{P}}{\Delta V} = \frac{d\mathbf{P}}{dV}$
- Traction: $\mathbf{T}_n = \lim_{\Delta S \to 0} \frac{\Delta \mathbf{F}}{\Delta S} = \frac{d\mathbf{F}}{dS}$
- Stress : x, y, z components of traction

2.2. Definition of Stress



2.3. Cauchy's Tetrahedron and Relation





- Equilibrium Condition

$$T_i^n \Delta S - t_i^1 \Delta S_1 - t_i^2 \Delta S_2 - t_i^3 \Delta S_3 + b_i \Delta V - \rho \frac{d^2 u_i}{dt^2} \Delta V = 0$$

$$T_i^n = t_i^1 \frac{\Delta S_1}{\Delta S} + t_i^2 \frac{\Delta S_2}{\Delta S} + t_i^3 \frac{\Delta S_3}{\Delta S} - b_i \frac{\Delta V}{\Delta S} + \rho \frac{d^2 u_i}{dt^2} \frac{\Delta V}{\Delta S}$$

$$\Delta V = \frac{1}{3} h \Delta S = \frac{1}{3} OA \Delta S_1 = \frac{1}{3} OB \Delta S_2 = \frac{1}{3} OC \Delta S_3$$

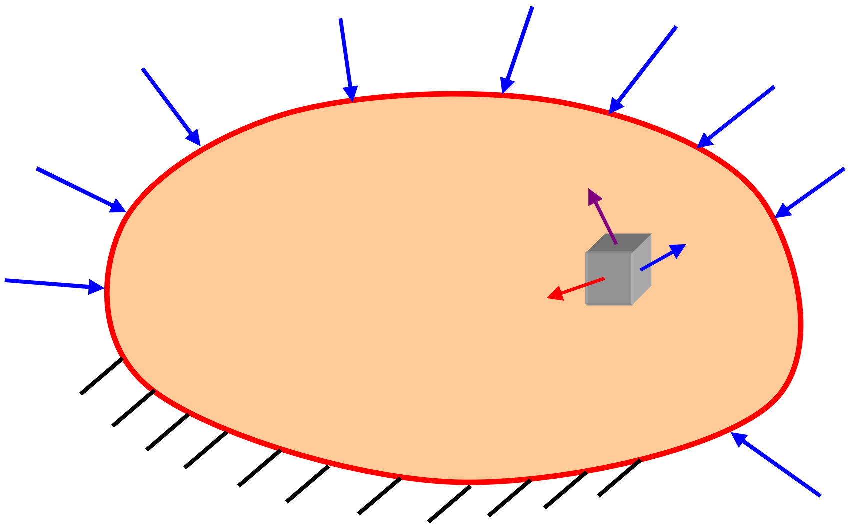
$$\frac{\Delta S_1}{\Delta S} = \frac{h}{OA} = \cos(n, x_1) = n_1, \quad \frac{\Delta S_2}{\Delta S} = \frac{h}{OB} = \cos(n, x_2) = n_2, \quad \frac{\Delta S_3}{\Delta S} = \frac{h}{OC} = \cos(n, x_3) = n_3$$

- **Cauchy's relation**

By the definition of stress $t_i^j = \sigma_{ji}$ and as $\Delta V \to 0$

$$T_i^n = t_i^1 n_1 + t_i^2 n_2 + t_i^3 n_3 \to T_i^n = \sigma_{ji} n_j$$

2.4. Equilibrium Equation - Integral Approach



- Force Equilibrium: $\sum F_i = 0$ for i=1, 2, 3

$$\int_S T_i dS + \int_V b_i dV - \int_V \rho \frac{\partial^2 u_i}{\partial t^2} dV = 0 \quad \text{or} \quad \int_S \mathbf{T} dS + \int_V \mathbf{b} dV - \int_V \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} dV = 0$$

By the divergence theorem, and the Cauchy's relationship

$$\int_S T_i dS = \int_S \sigma_{ji} n_j dS = \int_V \frac{\partial \sigma_{ji}}{\partial x_j} dV$$

Therefore, the force equilibrium equation becomes

$$\int_V \frac{\partial \sigma_{ji}}{\partial x_j} dV + \int_V b_i dV - \int_V \rho \frac{\partial^2 u_i}{\partial t^2} dV = \int_V \left(\frac{\partial \sigma_{ji}}{\partial x_j} + b_i - \rho \frac{\partial^2 u_i}{\partial t^2}\right) dV = 0$$

Since the above equation should satisfy for all bodies under equilibrium, the integrand should vanish everywhere in the domain.

$$\frac{\partial \sigma_{ji}}{\partial x_j} + b_i - \rho \frac{\partial^2 u_i}{\partial t^2} = 0 \longrightarrow \sigma_{ji,j} + b_i - \rho \frac{\partial^2 u_i}{\partial t^2} = 0 \longrightarrow \nabla \cdot \boldsymbol{\sigma} + \mathbf{b} - \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = 0$$

- Moment Equilibrium: $\sum M_i = 0$ for i=1, 2, 3

$$\int_S \mathbf{x} \times \mathbf{T} dS + \int_V \mathbf{x} \times \mathbf{b} dV + \int_V \mathbf{m} dV - \int_V \rho \mathbf{x} \times \frac{\partial^2 \mathbf{u}}{\partial t^2} dV = 0$$ or the indicial form becomes

$$\int_S e_{imn} x_m T_n dS + \int_V e_{imn} x_m b_n dV + \int_V m_i dV - \int_V \rho e_{imn} x_m \frac{\partial^2 u_n}{\partial t^2} dV = 0$$

$$\int_S e_{imn} x_m T_n dS = \int_S e_{imn} x_m \sigma_{jn} n_j dS = \int_V \frac{\partial e_{imn} x_m \sigma_{jn}}{\partial x_j} dV = \int_V e_{imn} (\frac{\partial x_m}{\partial x_j} \sigma_{jn} + x_m \frac{\partial \sigma_{jn}}{\partial x_j}) dV =$$

$$\int_V e_{imn} (\delta_{mj} \sigma_{jn} + x_m \frac{\partial \sigma_{jn}}{\partial x_j}) dV = \int_V e_{imn} (\sigma_{mn} + x_m \frac{\partial \sigma_{jn}}{\partial x_j}) dV$$

$$\int_V e_{imn} (\sigma_{mn} + x_m \frac{\partial \sigma_{jn}}{\partial x_j}) dV + \int_V e_{imn} x_m b_n dV + \int_V m_i dV - \int_V \rho e_{imn} x_m \frac{\partial^2 u_n}{\partial t^2} dV = 0$$

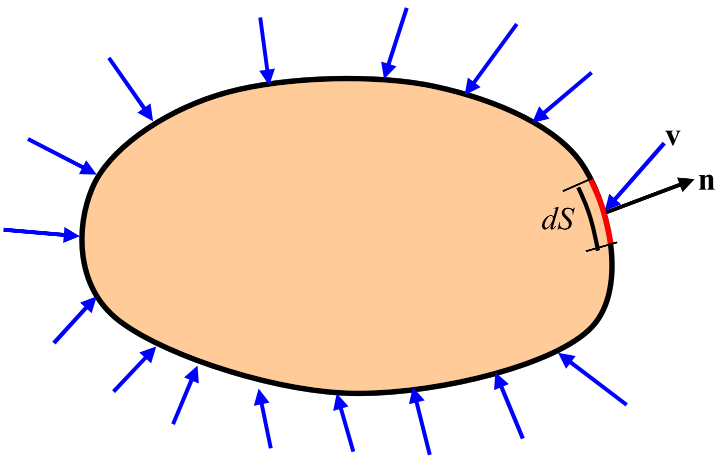
$$\int_V (e_{imn} \sigma_{mn} + m_i) dV + \int_V e_{imn} x_m (b_n + \frac{\partial \sigma_{jn}}{\partial x_j} - \rho \frac{\partial^2 u_n}{\partial t^2}) dV = 0$$

$$\int_V (e_{imn} \sigma_{mn} + m_i) dV = 0 \rightarrow e_{imn} \sigma_{mn} + m_i = 0$$

In case there is no body moment, $e_{imn}\sigma_{mn} = 0$

$$e_{imn}\sigma_{mn} = 0 \rightarrow \begin{cases} e_{1mn}\sigma_{mn} = 0 \rightarrow \sigma_{23} = \sigma_{32} \\ e_{2mn}\sigma_{mn} = 0 \rightarrow \sigma_{13} = \sigma_{31} \\ e_{3mn}\sigma_{mn} = 0 \rightarrow \sigma_{12} = \sigma_{21} \end{cases}$$

2.5. Conservation and Potential Problems



- Conservation in General

  Only the vector component normal to the surface (or boundary) flows into or out the volume. The tangential component just flows along the boundary without any effect on the vector field in the volume. Therefore, the conservation of the vector field is expressed as:

$$-\int_S \mathbf{v} \cdot \mathbf{n} dS + \int_V f dV = 0$$

The minus sign implies that in-flow direction is taken as the positive direction. As the outward normal vector always points outside of the volume, the inflow direction should be opposite to the outward normal vector of the surface.

– By divergence theorem,

$$-\int_S \mathbf{v}\cdot\mathbf{n}dS = -\int_V \nabla\cdot\mathbf{v}dV \quad \text{where} \quad \nabla = (\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}) = (\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\frac{\partial}{\partial x_3}).$$

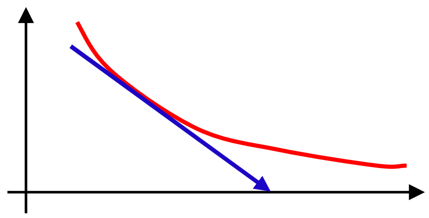
$$-\int_S \mathbf{v}\cdot\mathbf{n}dS + \int_V fdV = -\int_V \nabla\cdot\mathbf{v}dV + \int_V fdV = \int_V (-\nabla\cdot\mathbf{v} + f)dV = 0$$

– Since the integral equation should hold for all systems,

$$-\nabla\cdot\mathbf{v} + f = 0$$

- Potential Problems
  – In a potential problem, the vector field of a system is defined by a gradient of a scalar function referred to as a potential function

$$\mathbf{v} = -\mathbf{k}\cdot\nabla\Phi$$



– The famous **Laplace equation** for a conservative system.

$$-\nabla\cdot\mathbf{v} + f = \nabla\cdot(\mathbf{k}\cdot\nabla\Phi) + f = 0$$

– the system properties are homogeneous and isotropic, $\mathbf{v} = -k\nabla\Phi$

$$-\nabla\cdot\mathbf{v} + f = \nabla\cdot(k\nabla\Phi) + f = k\nabla^2\Phi + f = 0 \quad \text{or}$$

$$k(\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2}) + f = k\frac{\partial^2\Phi}{\partial x_i \partial x_i} + f = 0$$

2.6. Equilibrium and Potential Problems

- Equilibrium
  - Force Equilibrium: $\sum F_x = \sum F_y = \sum F_z = 0$ or $\sum \mathbf{F} = 0$

$$\int_S \mathbf{T} dS + \int_V \mathbf{b} dV = 0 \quad \text{or} \quad \int_S T_i dS + \int_V b_i dV = 0 \quad \text{for } i = 1,2,3$$

  - Suppose $\mathbf{T} = \boldsymbol{\sigma} \cdot \mathbf{n}$ or $T_i = \sum_{j=1}^{3} \sigma_{ij} n_j = \boldsymbol{\sigma}_i \cdot \mathbf{n}$

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\sigma}_1 \\ \boldsymbol{\sigma}_2 \\ \boldsymbol{\sigma}_3 \end{bmatrix}, \quad \mathbf{n} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

  - Divergence Theorem

$$\begin{aligned} \int_S T_i dS + \int_V b_i dV &= \int_S \boldsymbol{\sigma}_i \cdot \mathbf{n} dS + \int_V b_i dV = \int_V \nabla \cdot \boldsymbol{\sigma}_i dV + \int_V b_i dV \\ &= \int_V (\nabla \cdot \boldsymbol{\sigma}_i + b_i) dV = 0 \end{aligned} \qquad \text{for } i = 1,2,3$$

  - The integral equation should hold for all systems in equilibrium.

$$\nabla \cdot \boldsymbol{\sigma}_i + b_i = 0 \quad \text{for } i = 1, 2, 3$$

  - Moment Equilibrium: $\sum M_i = 0$ for i=1, 2, 3 or $\sum \mathbf{M} = 0$

$$\int_S \mathbf{x} \times \mathbf{T} dS + \int_V \mathbf{x} \times \mathbf{b} dV + \int_V \mathbf{m} dV = 0 \quad \text{or} \quad \sigma_{23} = \sigma_{32} \ , \ \sigma_{13} = \sigma_{31} \ , \ \sigma_{12} = \sigma_{21}$$

- Potential Problems
  - In case elasticity problems are potential problems

$$\boldsymbol{\sigma}_i = \mathbf{C}^i \cdot \nabla u_i \quad \text{or} \quad \sigma_{ij} = C^i_{jk} \frac{\partial u_i}{\partial x_k}$$

where repeated index i does not indicate the summation. However, in general, $\sigma_{ij} = C^i_{jk} \frac{\partial u_i}{\partial x_k} \neq \sigma_{ji} = C^j_{ik} \frac{\partial u_j}{\partial x_k}$ because u_i and u_j are independent potential functions. Therefore, to maintain symmetry condition of stress, each stress component should be a function of the gradients of all components of the potential functions, and furthermore the material properties should be defined as a fourth order tensor rather than a second order tensor:

$$\boldsymbol{\sigma} = \mathbf{C} : \nabla \mathbf{u} \quad \text{or} \quad \sigma_{ij} = C_{ijkl} \frac{\partial u_k}{\partial x_l} \quad \text{and} \quad C_{ijkl} = C_{jikl}$$

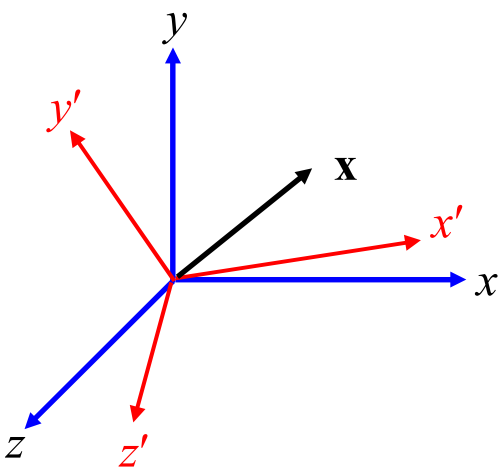
– In case $C_{ijkl} = C_{ijlk}$

$$\sigma_{ij} = C_{ijkl}\frac{\partial u_k}{\partial x_l} = \frac{1}{2}(C_{ijkl}\frac{\partial u_k}{\partial x_l} + C_{ijlk}\frac{\partial u_k}{\partial x_l}) = \frac{1}{2}(C_{ijkl}\frac{\partial u_k}{\partial x_l} + C_{ijkl}\frac{\partial u_l}{\partial x_k})$$
$$= C_{ijkl}\frac{1}{2}(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}) = C_{ijkl}\varepsilon_{kl}$$

– Equilibum equation in terms of the potential functions: $\nabla\cdot(\mathbf{C}:\nabla\mathbf{u}) + \mathbf{b} = 0$

2.7. Rotation of Axis and Stress

- Suppose a new primed coordinate system is defined, and we want to expresss each component of a vector in the primed coordinate system.



- Rotation of Axis

$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 = x'_1\mathbf{e}'_1 + x'_2\mathbf{e}'_2 + x'_3\mathbf{e}'_3$

$x_i\mathbf{e}_i = x'_k\mathbf{e}'_k \rightarrow \mathbf{e}'_j\cdot\mathbf{e}'_k x'_k = \mathbf{e}'_j\cdot\mathbf{e}_i x_i \rightarrow \delta_{kj}x'_k = \mathbf{e}'_j\cdot\mathbf{e}_i x_i \rightarrow x'_j = \mathbf{e}'_j\cdot\mathbf{e}_i x_i$

$x'_j = \beta_{ji}x_i$ where $\beta_{ji} = \mathbf{e}'_j\cdot\mathbf{e}_i = \cos(x'_j, x_i)$ or in tensor form $\mathbf{x}' = \boldsymbol{\beta}\mathbf{x}$

$x_i\mathbf{e}_i = x'_k\mathbf{e}'_k \rightarrow \mathbf{e}_j\cdot\mathbf{e}_i x_i = \mathbf{e}_j\cdot\mathbf{e}'_k x'_k \rightarrow \delta_{ji}x_i = \mathbf{e}_j\cdot\mathbf{e}'_k x'_k \rightarrow x_j = \mathbf{e}'_k\cdot\mathbf{e}_j x'_k = \beta_{kj}x'_k \rightarrow \mathbf{x} = \boldsymbol{\beta}^T\mathbf{x}'$

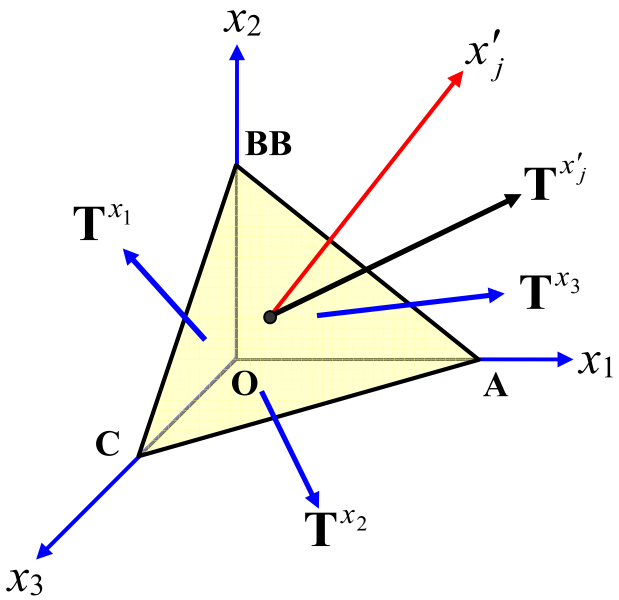
- Orthogonality

$x_j = \beta_{kj}x'_k = \beta_{kj}\beta_{ki}x_i \rightarrow (\delta_{ij} - \beta_{kj}\beta_{ki})x_i = 0 \rightarrow \beta_{kj}\beta_{ki} = \delta_{ij}$

$\beta_{ki}\beta_{kj} = \delta_{ij} \rightarrow \boldsymbol{\beta}^T\boldsymbol{\beta} = \mathbf{I} \rightarrow \boldsymbol{\beta}^{-1} = \boldsymbol{\beta}^T$

- Diffential Operator

$$\frac{\partial()}{\partial x_i} = \frac{\partial()}{\partial x'_k}\frac{\partial x'_k}{\partial x_i} = \frac{\partial()}{\partial x'_k}\frac{\partial \beta_{km}x_m}{\partial x_i} = \beta_{km}\frac{\partial()}{\partial x'_k}\delta_{mi} = \beta_{ki}\frac{\partial()}{\partial x'_k}$$



- Tractions in the directions of the primed (new) axes in the original CS.

  - Stress in each coordinate system by the deficition: $\sigma_{ij} = T_{x_j}^{x_i}$, $\sigma'_{ij} = T_{x'_j}^{x'_i}$

  - Cauchy's relation: $T_{x_i}^{x'_j} = T_{x_i}^{x_1} n_1^{x'_j} + T_{x_i}^{x_2} n_2^{x'_j} + T_{x_i}^{x_3} n_3^{x'_j} \rightarrow T_{x_i}^{x'_j} = \sigma_{ki} n_k^{x'_j} = \sigma_{ki}\beta_{jk}$

  - Normal vector: $\mathbf{n}^{x'_j} = \mathbf{e}'_j = n_1^{x'_j}\mathbf{e}_1 + n_2^{x'_j}\mathbf{e}_2 + n_3^{x'_j}\mathbf{e}_3 = n_k^{x'_j}\mathbf{e}_k \rightarrow n_k^{x'_j} = \mathbf{e}'_j \cdot \mathbf{e}_k = \cos(x'_j, x_k) = \beta_{jk}$

  The above tractions are given in the original CS, and thus each traction vector should be transformed to represent the stress in the primed CS.

- Stress in the primed coordinate system

$$\sigma'_{mk} = T_{x'_k}^{x'_m} = \beta_{kp} T_{x_p}^{x'_m} = \beta_{kp}\sigma_{qp}\beta_{mq} = \beta_{mq}\sigma_{qp}\beta_{kp} \rightarrow \boldsymbol{\sigma}' = \boldsymbol{\beta}\boldsymbol{\sigma}\boldsymbol{\beta}^T$$

$$\underline{\underline{\boldsymbol{\sigma} = \boldsymbol{\beta}^T\boldsymbol{\sigma}'\boldsymbol{\beta} \rightarrow \sigma_{ij} = \beta_{ki}\sigma'_{kl}\beta_{lj}}}$$

- Equilibrium Equation in primed coordinate system

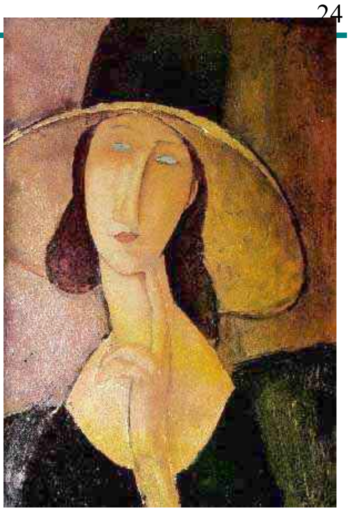
  - Stress: $\sigma_{ji} = \beta_{kj}\sigma'_{kl}\beta_{li}$, Body force: $b_i = \beta_{li} b'_l$, Diplacement: $u_i = \beta_{li} u'_l$

$$\frac{\partial \beta_{kj}\sigma'_{kl}\beta_{li}}{\partial x'_m}\beta_{mj} + \beta_{li} b'_l = \frac{\partial^2 \beta_{li} u'_l}{\partial t^2} \rightarrow \frac{\partial \sigma'_{kl}}{\partial x'_m}\beta_{kj}\beta_{li}\beta_{mj} + \beta_{li} b'_l = \beta_{li}\frac{\partial^2 u'_l}{\partial t^2} \rightarrow$$

$$\frac{\partial \sigma'_{kl}}{\partial x'_m}\beta_{li}\delta_{km} + \beta_{li} b'_l = \beta_{li}\frac{\partial^2 u'_l}{\partial t^2} \rightarrow \beta_{li}\left(\frac{\partial \sigma'_{kl}}{\partial x'_k} + b'_l - \frac{\partial^2 u'_l}{\partial t^2}\right) = 0$$

$$\frac{\partial \sigma'_{kl}}{\partial x'_k} + b'_l = \frac{\partial^2 u'_l}{\partial t^2}$$

  - The equilibrium equation is independent of a selection of coordinate system.



Chapter 3

Displacement and Strain

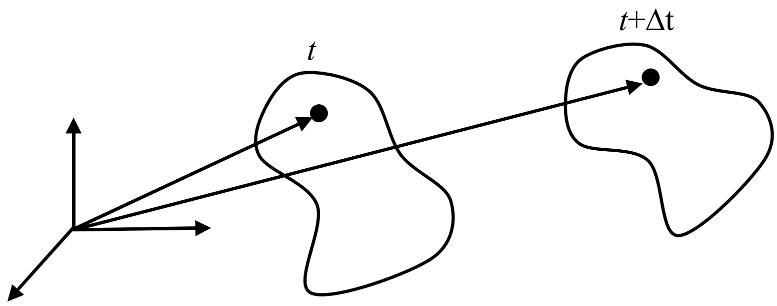
3.0. Kinematic Description

- **Kinematics?**

  A subdivision of classical mechanics concerned with the geometrically possible motion of a body or system of bodies without consideration of the forces involved (i.e., causes and effects of the motions).

- **Kinematic Description**

  Description of the **spatial position** of bodies or systems of material particles, the rate at which the particles are moving (**velocity**), and the rate at which their velocity is changing (**acceleration**)



Do the wind velocity and the velocity of a car have the same physical meaning?

- **How to describe the traffic condition of Olympic Highway?**

- Method I: Drive your own car on Olympic highway, and measure the velocity of your car with time. If you differentiate the measured velocity with respect to time, the acceleration of your car can be obtained.

$$\left.\frac{\partial \mathbf{V}(\mathrm{car},t)}{\partial t}\right|_{\text{car fixed}} = \lim_{\Delta t \to 0} \frac{\mathbf{V}(t+\Delta t)-\mathbf{V}(t)}{\Delta t} = \mathbf{a}$$

If the velocity of every car on the highway is measured and reported to you for all time, then you have a complete description of the traffic condition of the Olympic highway. No difficulty is encountered in formulating problems with this kinematic description (solid mechanics).

- Method II: Sit down at any location of your choice near the highway and observe and record the velocities of cars passing in front of you. If the measurement is taken place at every location on the highway, then you also have a complete description of the traffic condition of the Olympic highway. If you differentiate the recorded velocity at a location, the true acceleration defined by Sir Isaac Newton cannot be obtained. This is because you just calculated change of velocities of two cars passing though you at a specification location at two different time.

$$\left.\frac{\partial \mathbf{V}(\mathbf{x},t)}{\partial t}\right|_{\mathbf{x}\text{ fixed}} = \lim_{\Delta t \to 0} \frac{\mathbf{V}(t+\Delta t)-\mathbf{V}(t)}{\Delta t} \neq \mathbf{a}$$

Notice that $V(t+\Delta t)$ and $V(t)$ are velocoities of two different cars measured at different times at the same spatial location. The true acceleration becomes (out of scope of this class)

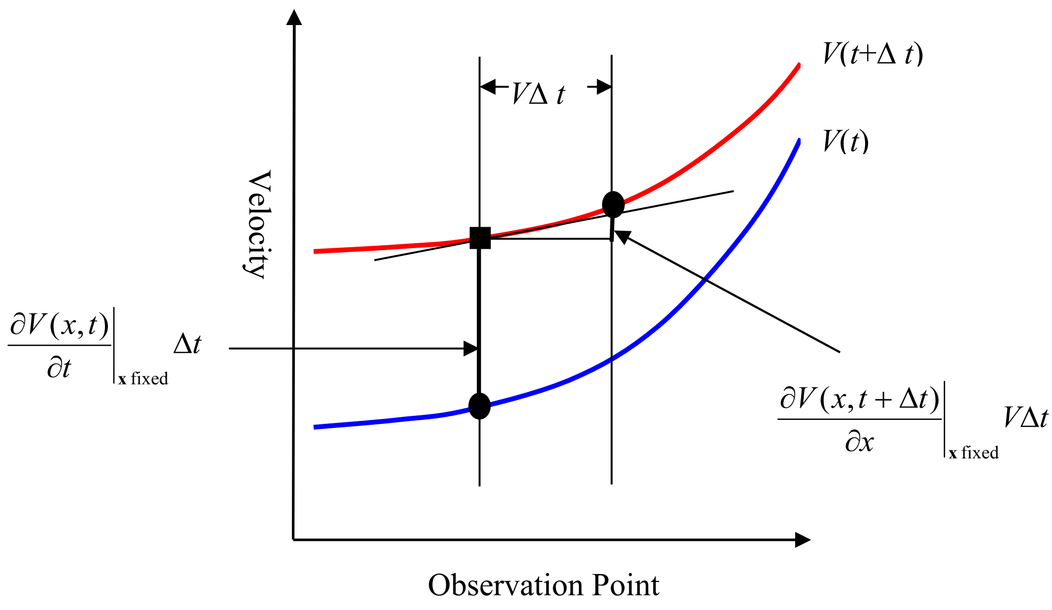
$$\mathbf{a} = \left.\frac{\partial \mathbf{V}(\mathbf{x},t)}{\partial t}\right|_{\mathbf{x}\text{ fixed}} + \nabla\mathbf{V}\cdot\mathbf{V}$$

This method is very convenient way to TBS, but results in a very, very, very (…) complicate situation in solving physical problems with this kinematic description (fluid mechanics).

Total change in the velocity of a fixed material particle

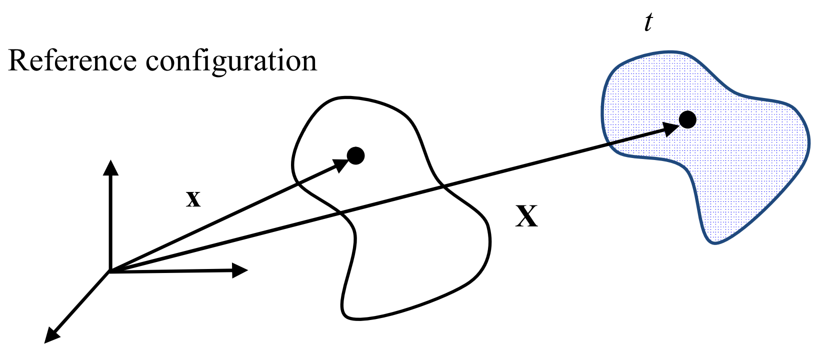
$$\left.\Delta V(x,t)\right|_{\text{Material fixed}} = \left.\frac{\partial V(x,t)}{\partial t}\right|_{\mathbf{x}\text{ fixed}} \Delta t + \frac{\partial V(x,t+\Delta t)}{\partial x} V(x,t)\Delta t$$

$$a = \lim_{\Delta t \to 0} \left.\frac{\Delta V(x,t)}{\Delta t}\right|_{\text{Material fixed}} = \left.\frac{\partial V(x,t)}{\partial t}\right|_{\mathbf{x}\text{ fixed}} + \frac{\partial V(x,t)}{\partial x} V(x,t)$$



- **Referential (Lagrangian) Description: Method I**

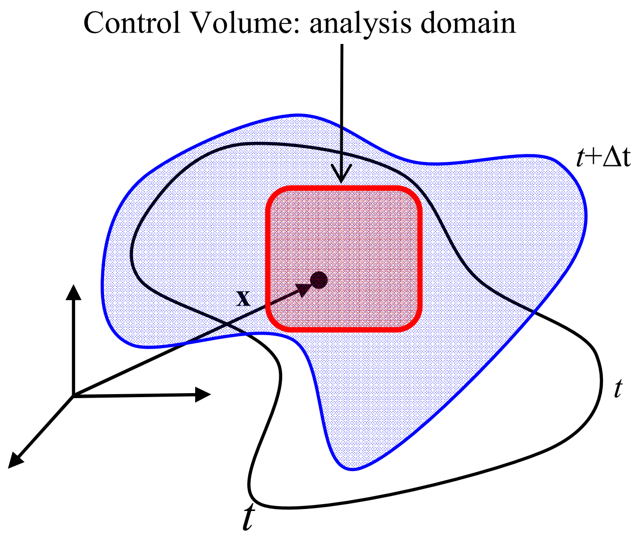
  Independent variables are the position **x** of the material particle in an arbitrarily chosen reference configuration, and the time *t*. In elasticity, the reference configuration is usually chosen to be the natural or unstrained position (known). When the reference configuration is chosen to be the actual initial configuration at $t=0$, the referential description is called the Lagrangian description. The reference position **x** is used for a label (or name) for the material particle occupying the position **x** in the reference configuration. All variables are considered as functions of **x**. Notice that **x** denotes the material particle occupying the position x in the reference configuration, not just a spatial point.



$$\mathbf{X} = \mathbf{X}(\mathbf{x}, t)$$

- **Spatial (Eulerian) Description: Method II**

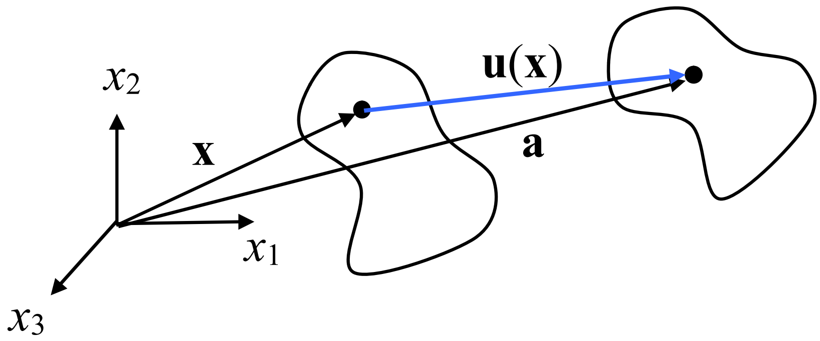
The independent variables are the spatial position vectors **x**. The spatial description fixes attention on a given region of space (control volume) instead of on a given body of matter. The material particles occupying a fixed position change for different time. It is the description most used in fluid mechanics, often called the Eulerian description, which gives us "Navier-Stokes equation", the most notorious partial differential equation in the engineering field.



3.1. Displacement and Strain

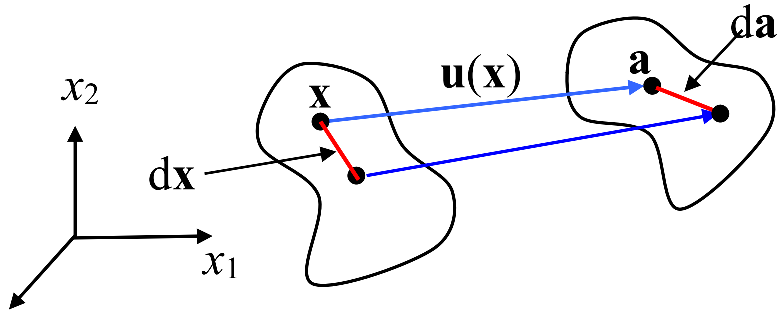
- Displacement is defined as the change of the spatial position of a fixed material particle.

- Displacement



$$\mathbf{u}(\mathbf{x}) = \mathbf{a}(\mathbf{x}) - \mathbf{x} \quad \text{or} \quad \mathbf{u}(\mathbf{a}) = \mathbf{a} - \mathbf{x}(\mathbf{a})$$

- Strain



$$ds_0^2 = dx_1^2 + dx_2^2 + dx_3^2 = dx_i dx_i \quad , \quad ds^2 = da_1^2 + da_2^2 + da_3^2 = da_i da_i$$

$$da_i = \frac{\partial a_i}{\partial x_k} dx_k \quad , \quad dx_i = \frac{\partial x_i}{\partial a_k} da_k$$

$$ds_0^2 = dx_i dx_i = \delta_{ij} dx_i dx_j = \delta_{ij} \frac{\partial x_i}{\partial a_k} da_k \frac{\partial x_j}{\partial a_l} da_l = \delta_{ij} \frac{\partial x_i}{\partial a_k} \frac{\partial x_j}{\partial a_l} da_k da_l$$

$$ds^2 = da_i da_i = \delta_{ij} da_i da_j = \delta_{ij} \frac{\partial a_i}{\partial x_k} dx_k \frac{\partial a_j}{\partial x_l} dx_l = \delta_{ij} \frac{\partial a_i}{\partial x_k} \frac{\partial a_j}{\partial x_l} dx_k dx_l$$

$$ds^2 - ds_0^2 = \delta_{ij} \frac{\partial a_i}{\partial x_k} \frac{\partial a_j}{\partial x_l} dx_k dx_l - \delta_{ij} dx_i dx_j = \delta_{\alpha\beta} \frac{\partial a_\alpha}{\partial x_i} \frac{\partial a_\beta}{\partial x_j} dx_i dx_j - \delta_{ij} dx_i dx_j$$

$$= (\delta_{\alpha\beta} \frac{\partial a_\alpha}{\partial x_i} \frac{\partial a_\beta}{\partial x_j} - \delta_{ij}) dx_i dx_j = 2E_{ij} dx_i dx_j$$

E_{ij} is called as **Green's strain.**

$$ds^2 - ds_0^2 = \delta_{ij} da_i da_j - \delta_{ij} \frac{\partial x_i}{\partial a_k} \frac{\partial x_j}{\partial a_l} da_k da_l = \delta_{ij} da_i da_j - \delta_{\alpha\beta} \frac{\partial x_\alpha}{\partial a_i} \frac{\partial x_\beta}{\partial a_j} da_i da_j$$

$$= (\delta_{ij} - \delta_{\alpha\beta} \frac{\partial x_\alpha}{\partial a_i} \frac{\partial x_\beta}{\partial a_j}) da_i da_j = 2 e_{ij} da_i da_j$$

e_{ij} is called as **Almansi strain**.

- Green's Strain in terms of displacement

$$E_{ij} = \frac{1}{2}(\delta_{\alpha\beta} \frac{\partial a_\alpha}{\partial x_i} \frac{\partial a_\beta}{\partial x_j} - \delta_{ij}) = \frac{1}{2}(\delta_{\alpha\beta} \frac{\partial (x_\alpha + u_\alpha)}{\partial x_i} \frac{\partial (x_\beta + u_\beta)}{\partial x_j} - \delta_{ij})$$

$$= \frac{1}{2}\delta_{\alpha\beta}(\delta_{\alpha i} + \frac{\partial u_\alpha}{\partial x_i})(\delta_{\beta j} + \frac{\partial u_\beta}{\partial x_j}) - \frac{1}{2}\delta_{ij}$$

$$= \frac{1}{2}(\delta_{\alpha\beta}\delta_{\alpha i}\delta_{\beta j} + \delta_{\alpha\beta}\delta_{\alpha i}\frac{\partial u_\beta}{\partial x_j} + \delta_{\alpha\beta}\delta_{\beta j}\frac{\partial u_\alpha}{\partial x_i} + \delta_{\alpha\beta}\frac{\partial u_\alpha}{\partial x_i}\frac{\partial u_\beta}{\partial x_j} - \delta_{ij})$$

$$= \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_\alpha}{\partial x_i}\frac{\partial u_\alpha}{\partial x_j})$$

$$E_{11} = \frac{\partial u_1}{\partial x_1} + \frac{1}{2}(\frac{\partial u_\alpha}{\partial x_1}\frac{\partial u_\alpha}{\partial x_1}) = \frac{\partial u_1}{\partial x_1} + \frac{1}{2}[(\frac{\partial u_1}{\partial x_1})^2 + (\frac{\partial u_2}{\partial x_1})^2 + (\frac{\partial u_3}{\partial x_1})^2]$$

$$E_{22} = \frac{\partial u_2}{\partial x_2} + \frac{1}{2}(\frac{\partial u_\alpha}{\partial x_1}\frac{\partial u_\alpha}{\partial x_1}) = \frac{\partial u_2}{\partial x_2} + \frac{1}{2}[(\frac{\partial u_1}{\partial x_2})^2 + (\frac{\partial u_2}{\partial x_2})^2 + (\frac{\partial u_3}{\partial x_2})^2]$$

$$E_{33} = \frac{\partial u_3}{\partial x_3} + \frac{1}{2}(\frac{\partial u_\alpha}{\partial x_3}\frac{\partial u_\alpha}{\partial x_3}) = \frac{\partial u_3}{\partial x_3} + \frac{1}{2}[(\frac{\partial u_1}{\partial x_3})^2 + (\frac{\partial u_2}{\partial x_3})^2 + (\frac{\partial u_3}{\partial x_3})^2]$$

$$E_{12} = \frac{1}{2}(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} + \frac{\partial u_\alpha}{\partial x_1}\frac{\partial u_\alpha}{\partial x_2}) = \frac{1}{2}(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}) + \frac{1}{2}(\frac{\partial u_1}{\partial x_1}\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_1}\frac{\partial u_3}{\partial x_2})$$

$$E_{13} = \frac{1}{2}(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3} + \frac{\partial u_\alpha}{\partial x_1}\frac{\partial u_\alpha}{\partial x_3}) = \frac{1}{2}(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}) + \frac{1}{2}(\frac{\partial u_1}{\partial x_1}\frac{\partial u_1}{\partial x_3} + \frac{\partial u_2}{\partial x_1}\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\frac{\partial u_3}{\partial x_3})$$

$$E_{32} = \frac{1}{2}(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} + \frac{\partial u_\alpha}{\partial x_3}\frac{\partial u_\alpha}{\partial x_2}) = \frac{1}{2}(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}) + \frac{1}{2}(\frac{\partial u_1}{\partial x_3}\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_3}\frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\frac{\partial u_3}{\partial x_2})$$

- Cauchy's infinitesimal strain tensor – Small deformation

$$\varepsilon_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}) \quad \text{for small displacement gradient} \left|\frac{\partial u_i}{\partial x_j}\right| << 1$$

In this case, $E_{ij} = e_{ij} = \varepsilon_{ij}$

- All kinds of strain are symmetric.

- Rigid body motion : $ds = ds_0$

  $ds^2 - ds_0^2 = 2E_{ij}dx_i dx_j = 2e_{ij}da_i da_j$ for d**x** and d**a**. $\leftrightarrow E_{ij} = e_{ij} \equiv 0$ for whole domain.

  For small strain, $\varepsilon_{ij} \equiv 0$.

- Engineering strain : $\gamma_{ij} = 2\varepsilon_{ij}$ for $i \neq j$

- Strain Component in primed coordinate system
  - Diplacement : $u_i = \beta_{ki} u'_k$
  - Strain

$$
\begin{aligned}
E_{ij} &= \frac{1}{2}\left(\frac{\partial \beta_{ki} u'_k}{\partial x_j} + \frac{\partial \beta_{kj} u'_k}{\partial x_i} + \frac{\partial \beta_{k\alpha} u'_k}{\partial x_i}\frac{\partial \beta_{l\alpha} u'_l}{\partial x_j}\right) \\
&= \frac{1}{2}\left(\frac{\partial \beta_{ki} u'_k}{\partial x'_m}\beta_{mj} + \frac{\partial \beta_{kj} u'_k}{\partial x'_m}\beta_{mi} + \frac{\partial \beta_{k\alpha} u'_k}{\partial x'_m}\beta_{mi}\frac{\partial \beta_{l\alpha} u'_l}{\partial x'_n}\beta_{nj}\right) \\
&= \frac{1}{2}\left(\frac{\partial \beta_{ki} u'_k}{\partial x'_m}\beta_{mj} + \frac{\partial \beta_{mj} u'_m}{\partial x'_k}\beta_{ki} + \delta_{kl}\frac{\partial u'_k}{\partial x'_m}\beta_{mi}\frac{\partial u'_l}{\partial x'_n}\beta_{nj}\right) \\
&= \frac{1}{2}\left(\beta_{ki}\frac{\partial u'_k}{\partial x'_m}\beta_{mj} + \beta_{mj}\frac{\partial u'_m}{\partial x'_k}\beta_{ki} + \beta_{mi}\frac{\partial u'_l}{\partial x'_m}\frac{\partial u'_l}{\partial x'_n}\beta_{nj}\right) \\
&= \frac{1}{2}\left(\beta_{ki}\frac{\partial u'_k}{\partial x'_m}\beta_{mj} + \beta_{mj}\frac{\partial u'_m}{\partial x'_k}\beta_{ki} + \beta_{ki}\frac{\partial u'_l}{\partial x'_k}\frac{\partial u'_l}{\partial x'_m}\beta_{mj}\right) \\
&= \beta_{ki}\frac{1}{2}\left(\frac{\partial u'_k}{\partial x'_m} + \frac{\partial u'_m}{\partial x'_k} + \frac{\partial u'_l}{\partial x'_k}\frac{\partial u'_l}{\partial x'_m}\right)\beta_{mj} = \beta_{ki}E'_{km}\beta_{mj}
\end{aligned}
$$

$$
\mathbf{E} = \boldsymbol{\beta}^T \mathbf{E}' \boldsymbol{\beta}, \quad \mathbf{E}' = \boldsymbol{\beta} \mathbf{E} \boldsymbol{\beta}^T
$$

3.2. Physical Meanings

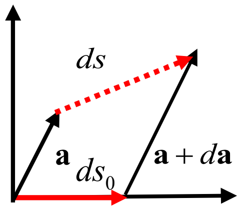
- Normal Strain

For x_1 direction, $(dx_1, 0, 0)$

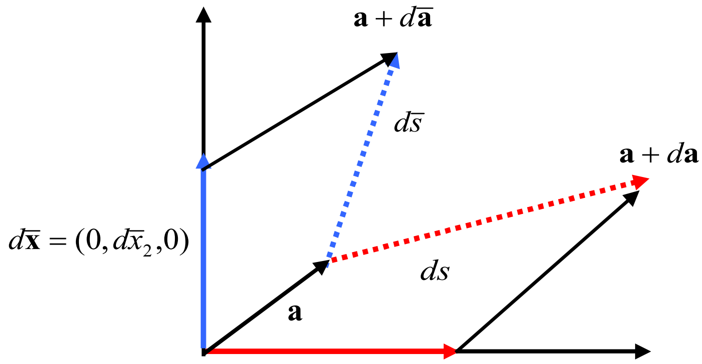
$$ds^2 - ds_0^2 = 2E_{ij}dx_i dx_j = 2E_{11}dx_1 dx_1 = 2E_{11}ds_0^2, \rightarrow ds = \sqrt{1+2E_{11}}\, ds_0$$

$$E_{11} = \frac{ds^2 - ds_0^2}{2ds_0^2} = \frac{ds - ds_0}{ds_0}\frac{ds + ds_0}{2ds_0}$$

Since $ds \approx ds_0$ for small strain, then $E_{11} = \varepsilon_{11} = \dfrac{ds - ds_0}{ds_0}$



- Shear Strain



$$d\mathbf{a}\cdot d\bar{\mathbf{a}} = ds d\bar{s}\cos\theta = da_k d\bar{a}_k = \frac{\partial a_k}{\partial x_m}dx_m \frac{\partial a_k}{\partial x_n}d\bar{x}_n$$

$$= \frac{\partial a_k}{\partial x_m}\frac{\partial a_k}{\partial x_n}dx_m d\bar{x}_n = \frac{\partial a_k}{\partial x_1}\frac{\partial a_k}{\partial x_2}dx_1 d\bar{x}_2 = 2E_{12}dx_1 d\bar{x}_2$$

$$\cos\theta = \frac{2E_{12}dx_1 d\bar{x}_2}{ds d\bar{s}}$$

$$ds = \sqrt{1+2E_{11}}\, ds_0 = \sqrt{1+2E_{11}}\, dx_1, \quad d\bar{s} = \sqrt{1+2E_{22}}\, d\bar{s}_0 = \sqrt{1+2E_{22}}\, d\bar{x}_2$$

$$\cos\theta = \sin(\alpha_{12}) = \frac{2E_{12}}{\sqrt{1+2E_{11}}\sqrt{1+2E_{22}}}$$

$$E_{12} = \frac{1}{2}\sin(\alpha_{12})\sqrt{1+2E_{11}}\sqrt{1+2E_{22}}$$

In case E_{11} , $E_{22} << 1$, $\quad E_{12} = \dfrac{1}{2}\sin(\alpha_{12}) \approx \dfrac{1}{2}\alpha_{12}$

3.3. Compatibility in 2-D Problems

- In case the strain field is given, does the unique displacement field possibly exist ?

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} \ , \ \varepsilon_{22} = \frac{\partial u_2}{\partial x_2} \ , \ \varepsilon_{33} = \frac{\partial u_3}{\partial x_3}$$

$$\varepsilon_{12} = \frac{1}{2}(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}) \ , \ \varepsilon_{13} = \frac{1}{2}(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}) \ , \ \varepsilon_{32} = \frac{1}{2}(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2})$$

- In case that displacement components are given, six independent strain components can be defined without any ambiguity.
- What if six independent strain components are given to define displacement of a body? Can we always calculate three independent displacement components uniquely?
- There are six strain components, but only three displacement components exist.
- We have to determine three unknowns using six equations, which generally is impossible to solve. Therefore, all the strain components are not independent.
- **Three additional relationships between strain components should be defined.**
- **We refer to the above conditions as the compatibility condition!**

- What conditions is required to do so ?

- Two dimensional Case :

$$\varepsilon_{33} = \varepsilon_{13} = \varepsilon_{23} = 0, \ \ \varepsilon_{11} = \frac{\partial u_1}{\partial x_1} \ , \varepsilon_{22} = \frac{\partial u_2}{\partial x_2} \ , \ \varepsilon_{12} = \frac{1}{2}(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1})$$

$$u_1 = \int \varepsilon_{11} dx_1 + g(x_2) \quad , \quad u_2 = \int \varepsilon_{22} dx_2 + f(x_1)$$

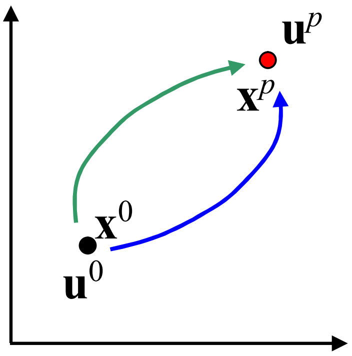
$$\varepsilon_{12} = \frac{1}{2}(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}) = \frac{1}{2}(\frac{\partial}{\partial x_2}\int \varepsilon_{11} dx_1 + \frac{\partial g(x_2)}{\partial x_2} + \frac{\partial}{\partial x_1}\int \varepsilon_{22} dx_2 + \frac{\partial f(x_1)}{\partial x_1})$$

$$\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} = \frac{1}{2}(\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2}) \ , \ \ \frac{\partial^2 \gamma_{12}}{\partial x_1 \partial x_2} = \frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2}$$

- The strain field can not be defined independently !!!

3.4. General Case

- Path Independent condition for displacement
  - The displacement field should be single-valued functions, which are independent of path.
  - This condition is related to the second axiom and the finiteness of strain energy.



$$u_j^p = u_j^0 + \int_C du_j = u_j^0 + \int_C \frac{\partial u_j}{\partial x_k} dx_k = u_j^0 + \frac{1}{2}\int_C \left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\right) dx_k + \frac{1}{2}\int_C \left(\frac{\partial u_j}{\partial x_k} - \frac{\partial u_k}{\partial x_j}\right) dx_k$$

$$= u_j^0 + \int_C \varepsilon_{jk} dx_k + \int_C \Omega_{jk} dx_k$$

$$\int_C \Omega_{jk} dx_k = \int_C \Omega_{jk} d(x_k - x_k^p) = (x_k^p - x_k^0)\Omega_{jk}^0 - \int_C (x_k - x_k^p) \frac{\partial \Omega_{jk}}{\partial x_m} dx_m$$

$$\frac{\partial \Omega_{jk}}{\partial x_m} = \frac{\partial}{\partial x_m}\frac{1}{2}\left(\frac{\partial u_j}{\partial x_k} - \frac{\partial u_k}{\partial x_j}\right) + \frac{1}{2}\frac{\partial^2 u_m}{\partial x_j \partial x_k} - \frac{1}{2}\frac{\partial^2 u_m}{\partial x_j \partial x_k}$$

$$= \frac{\partial}{\partial x_k}\frac{1}{2}\left(\frac{\partial u_j}{\partial x_m} + \frac{\partial u_m}{\partial x_j}\right) - \frac{\partial}{\partial x_j}\frac{1}{2}\left(\frac{\partial u_k}{\partial x_m} + \frac{\partial u_m}{\partial x_k}\right) = \frac{\partial \varepsilon_{mj}}{\partial x_k} - \frac{\partial \varepsilon_{km}}{\partial x_j}$$

$$u_j^p = u_j^0 + \int_C \varepsilon_{jk} dx_k + \int_C \Omega_{jk} dx_k$$

$$= u_j^0 + (x_k^p - x_k^0)\Omega_{jk}^0 + \int_C \left(\varepsilon_{jm} - (x_k - x_k^p)\left(\frac{\partial \varepsilon_{mj}}{\partial x_k} - \frac{\partial \varepsilon_{km}}{\partial x_j}\right)\right) dx_m$$

$$= u_j^0 + (x_k^p - x_k^0)\Omega_{jk}^0 + \int_C F_{jm} dx_m$$

- For the path independent condition

$$\frac{\partial F_{jn}}{\partial x_m} = \frac{\partial F_{jm}}{\partial x_n}$$

$$\frac{\partial}{\partial x_m}[\varepsilon_{jn} - (x_k - x_k^p)(\frac{\partial \varepsilon_{nj}}{\partial x_k} - \frac{\partial \varepsilon_{kn}}{\partial x_j})] = \frac{\partial}{\partial x_n}[\varepsilon_{jm} - (x_k - x_k^p)(\frac{\partial \varepsilon_{mj}}{\partial x_k} - \frac{\partial \varepsilon_{km}}{\partial x_j})]$$

$$\frac{\partial \varepsilon_{jn}}{\partial x_m} - \delta_{km}(\frac{\partial \varepsilon_{nj}}{\partial x_k} - \frac{\partial \varepsilon_{kn}}{\partial x_j}) - \frac{\partial \varepsilon_{jm}}{\partial x_n} + \delta_{kn}(\frac{\partial \varepsilon_{mj}}{\partial x_k} - \frac{\partial \varepsilon_{km}}{\partial x_j}) -$$
$$(x_k - x_k^p)[\frac{\partial^2 \varepsilon_{nj}}{\partial x_m \partial x_k} - \frac{\partial^2 \varepsilon_{kn}}{\partial x_m \partial x_j} - \frac{\partial^2 \varepsilon_{mj}}{\partial x_n \partial x_k} + \frac{\partial^2 \varepsilon_{km}}{\partial x_n \partial x_j}] = 0$$

$$\frac{\partial \varepsilon_{jn}}{\partial x_m} - (\frac{\partial \varepsilon_{nj}}{\partial x_m} - \frac{\partial \varepsilon_{mn}}{\partial x_j}) - \frac{\partial \varepsilon_{jm}}{\partial x_n} + (\frac{\partial \varepsilon_{mj}}{\partial x_n} - \frac{\partial \varepsilon_{nm}}{\partial x_j}) -$$
$$(x_k - x_k^p)[\frac{\partial^2 \varepsilon_{nj}}{\partial x_m \partial x_k} - \frac{\partial^2 \varepsilon_{kn}}{\partial x_m \partial x_j} - \frac{\partial^2 \varepsilon_{mj}}{\partial x_n \partial x_k} + \frac{\partial^2 \varepsilon_{km}}{\partial x_n \partial x_j}] = 0$$

$$\frac{\partial^2 \varepsilon_{nj}}{\partial x_m \partial x_k} - \frac{\partial^2 \varepsilon_{kn}}{\partial x_m \partial x_j} - \frac{\partial^2 \varepsilon_{mj}}{\partial x_n \partial x_k} + \frac{\partial^2 \varepsilon_{km}}{\partial x_n \partial x_j} = 0 \quad \text{(by E. Cesaro)}$$

- St.-Venant's Compatibility Equations

$$R_1 = \frac{\partial^2 \varepsilon_{22}}{\partial x_3^2} + \frac{\partial^2 \varepsilon_{33}}{\partial x_2^2} - 2\frac{\partial^2 \varepsilon_{23}}{\partial x_2 \partial x_3} = 0$$
$$R_2 = \frac{\partial^2 \varepsilon_{33}}{\partial x_1^2} + \frac{\partial^2 \varepsilon_{11}}{\partial x_3^2} - 2\frac{\partial^2 \varepsilon_{13}}{\partial x_1 \partial x_3} = 0$$
$$R_3 = \frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} - 2\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} = 0$$
$$U_1 = -\frac{\partial^2 \varepsilon_{11}}{\partial x_2 \partial x_3} + \frac{\partial}{\partial x_1}(-\frac{\partial \varepsilon_{23}}{\partial x_1} + \frac{\partial \varepsilon_{13}}{\partial x_2} + \frac{\partial \varepsilon_{12}}{\partial x_3}) = 0$$
$$U_2 = -\frac{\partial^2 \varepsilon_{22}}{\partial x_1 \partial x_3} + \frac{\partial}{\partial x_2}(\frac{\partial \varepsilon_{23}}{\partial x_1} - \frac{\partial \varepsilon_{13}}{\partial x_2} + \frac{\partial \varepsilon_{12}}{\partial x_3}) = 0$$
$$U_3 = -\frac{\partial^2 \varepsilon_{33}}{\partial x_1 \partial x_2} + \frac{\partial}{\partial x_3}(\frac{\partial \varepsilon_{23}}{\partial x_1} + \frac{\partial \varepsilon_{13}}{\partial x_2} - \frac{\partial \varepsilon_{12}}{\partial x_3}) = 0$$

We need only three equations, but six equations are resulted in. What happens?

- Bianchi formulas

$$\frac{\partial R_1}{\partial x_1}+\frac{\partial U_3}{\partial x_2}+\frac{\partial U_2}{\partial x_3}=0\ ,\ \frac{\partial U_3}{\partial x_1}+\frac{\partial R_2}{\partial x_2}+\frac{\partial U_1}{\partial x_3}=0\ ,\ \frac{\partial U_2}{\partial x_1}+\frac{\partial U_1}{\partial x_2}+\frac{\partial R_3}{\partial x_3}=0$$

The Bianchi formulas show that the St. Veanat's equations are not independent, but three compatibility equations out of six are dependent. Unfortunately, we still have one more issue on the selection of three independent relationships out of six!

- Dependence of Compatibility (K. Washizu, 1957)

Suppose one set of the compatibility equations are satisfied in the domain, while the other set are satisfied on the boundary. Then, the compatibility equations specified on the boundary are automatically satisfied in the domain. Therefore, only one set of the compatibility equations is required to be satisfied either in the domain or on the boundary.

- Case 1

$U_1=U_2=U_3=0$ in V and $R_1=R_2=R_3=0$ on S. By Bianchi condition

$\frac{\partial R_1}{\partial x}=\frac{\partial R_2}{\partial y}=\frac{\partial R_3}{\partial z}=0$ in $V \rightarrow R_1=R_2=R_3=0$ in V

- Case 2

$R_x=R_y=R_z=0$ in V and $U_x=U_y=U_z=0$ on S. By Bianchi condition

$\frac{\partial U_3}{\partial x_2}+\frac{\partial U_2}{\partial x_3}=0,\ \frac{\partial U_3}{\partial x_1}+\frac{\partial U_1}{\partial x_3}=0,\ \frac{\partial U_2}{\partial x_1}+\frac{\partial U_1}{\partial x_2}=0$ in V.

For arbitrary $F(x,y,z)$, $G(x,y,z)$ and $H(x,y,z)$

$$\begin{aligned} I &= \int_V [F(\frac{\partial U_3}{\partial x_2}+\frac{\partial U_2}{\partial x_3})+G(\frac{\partial U_3}{\partial x_1}+\frac{\partial U_1}{\partial x_3})+H(\frac{\partial U_2}{\partial x_1}+\frac{\partial U_1}{\partial x_2})dV \\ &= \int_S [(nG+mH)U_1+(nF+lH)U_2+(lG+mF)U_3]dS \\ &\quad -\int_V [U_1(\frac{\partial H}{\partial x_2}+\frac{\partial G}{\partial x_3})+U_2(\frac{\partial H}{\partial x_1}+\frac{\partial F}{\partial x_3})+U_3(\frac{\partial G}{\partial x_1}+\frac{\partial F}{\partial x_2})dV \\ &= 0 \end{aligned}$$

Since $F(x,y,z)$, $G(x,y,z)$ and $H(x,y,z)$ are arbitrary, $U_x=U_y=U_z\equiv 0$ in V.

Chapter 4

Constitutive Relations

4.1. Constitutive Law

- Generalized Hooke's Law

$$\sigma_{ij} = C_{ijkl}E_{kl} = C_{ijkl}\varepsilon_{kl}$$

- Governing Equations in solid mechanics
  - Equilibrium : $\sigma_{ij,j} + b_i = \rho \dfrac{\partial^2 u_i}{\partial t^2}$
  - Strain-displacement relation : $\varepsilon_{ij} = \dfrac{1}{2}(\dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i})$
  - Constitutive law : $\sigma_{ij} = C_{ijkl}E_{kl} = C_{ijkl}\varepsilon_{kl}$

- Equilibrium Equations in terms of displacement

$$C_{ijkl}\frac{\partial^2 u_k}{\partial x_j \partial x_l} + b_i = \rho \frac{\partial^2 u_i}{\partial t^2}$$

- Boundary conditions
  - Displacement BC : $u_i = \bar{u}_i$ on Γ_u
  - Traction BC : $T_i = \bar{T}_i$ on Γ_t

- Traction BC in terms of displacement

$$T_i = \sigma_{ij}n_j = C_{ijkl}\frac{\partial u_k}{\partial x_l}n_j = \bar{T}_i$$

4.2. Isotropic Material

- Isotropy ?

"Iso" means identical or same, and "tropy" means "tendency to change in response to a stimulus". "Isotropy" stands for identical tendency to change in response to a stimulus. Therefore, "an isotropic material" indicates a material exhibiting identical properties when measured along axes in all directions. In other words, the material properties are independent of the selection of axes.

- Characteristics of Elasticity Tensor C_{ijkl}
  - Total number of coefficient: 81 $(= 3 \times 3 \times 3 \times 3)$

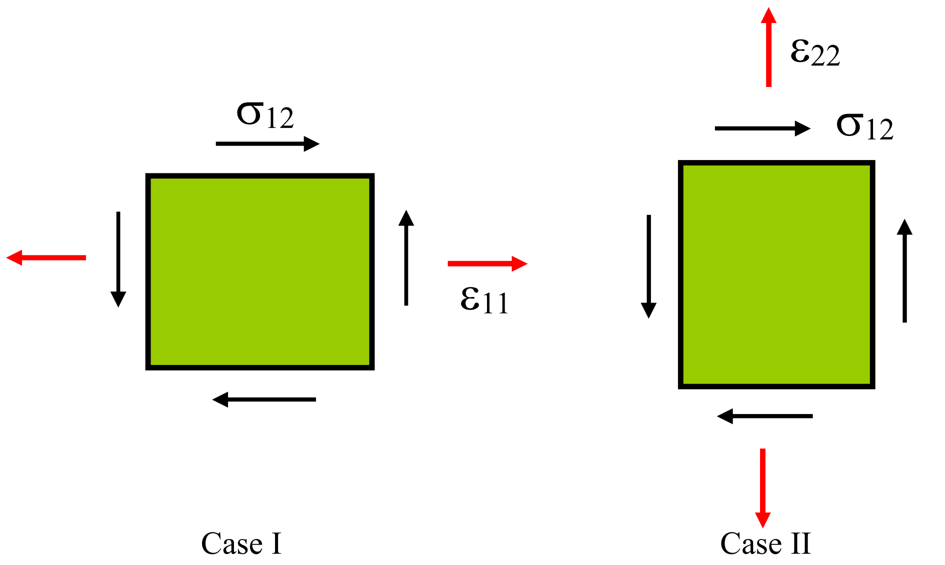
– Symmetry on σ and ε: $C_{ijkl} = C_{jikl}$, $C_{ijkl} = C_{ijlk}$ (36)

– Symmetry w.r.t *ij* and *kl* : $C_{ijkl} = C_{klij}$ (21 =(36-6)/2+6)

- Independent Relations

Rel.	Coefficient	# of Coeff.	# of Ind. Rel.
N-N	$C_{1111} = C_{2222} = C_{3333}$	3	1
N-O.N.	$C_{1122} = C_{2233} = C_{1133}$	3	1
S-S	$C_{1212} = C_{2323} = C_{1313}$	3	1
S-O.S	$C_{1213} = C_{2312} = C_{1323}$	3	1
N-S (S-N)	$C_{1112} = C_{1113} = C_{1123} =$ $C_{2212} = C_{2213} = C_{2223} =$ $C_{3312} = C_{3313} = C_{3323}$	9	1
Total		**21**	**5**

- Normal-Shear Relation = 0



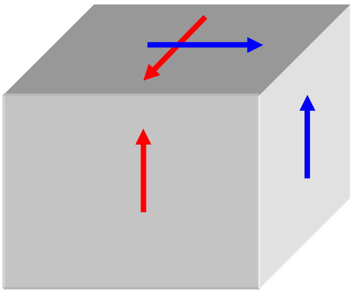
– Case I ($\varepsilon_{11} \neq 0$, $\varepsilon_{22} = \varepsilon_{33} = \varepsilon_{12} = \varepsilon_{13} = \varepsilon_{23} = 0$) $\rightarrow$ $\sigma_{12}^{I} = C_{1211}\varepsilon_{11}$

– Case II ($\varepsilon_{22} \neq 0$, $\varepsilon_{11} = \varepsilon_{33} = \varepsilon_{12} = \varepsilon_{13} = \varepsilon_{23} = 0$) $\rightarrow \sigma_{12}^{II} = C_{1222}\varepsilon_{22}$

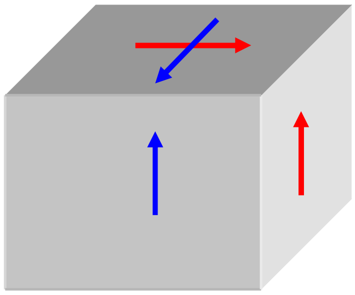
By isotropic condition, $\sigma_{12}^{I} = -\sigma_{12}^{II}$, for $\varepsilon_{11} = \varepsilon_{22}$.

$$(C_{1211} + C_{1222})\varepsilon_{11} = 0 \rightarrow 2C_{1211}\varepsilon_{11} = 0 \rightarrow C_{1211} = 0$$

- Shear-Other Shear Relation



Case I



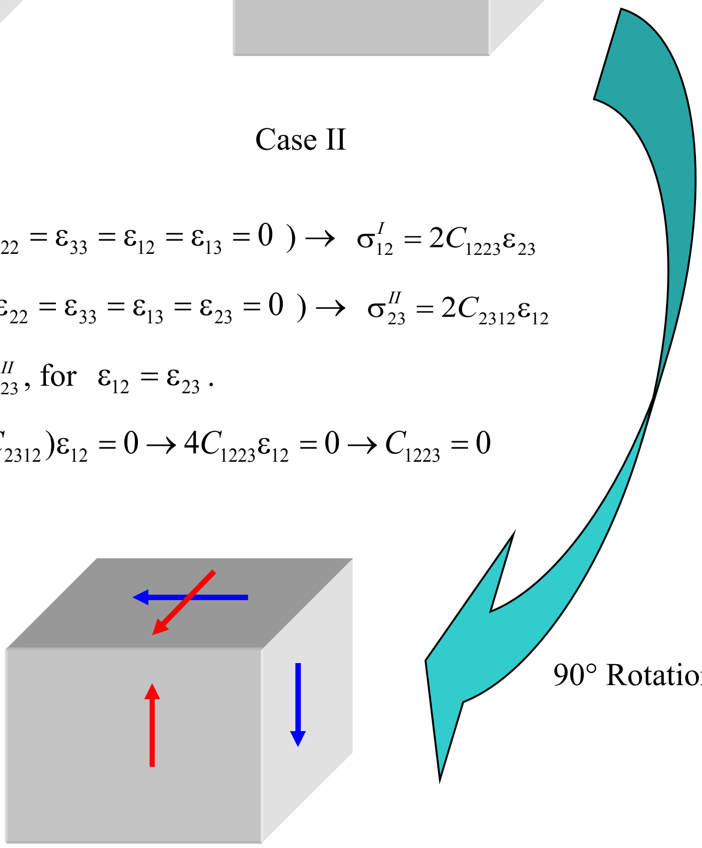
Case II

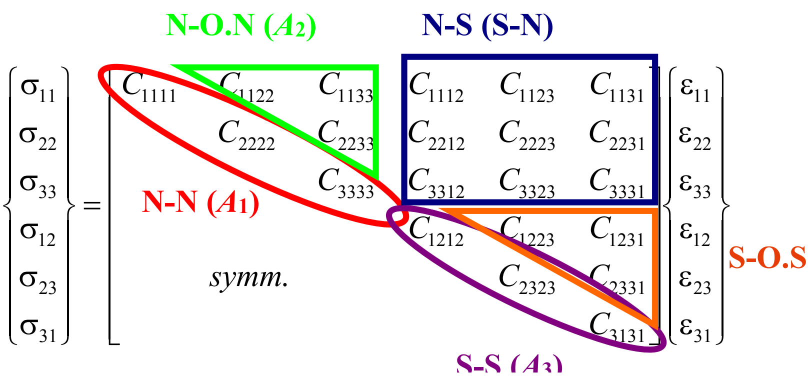
– Case I ($\varepsilon_{23} \neq 0$, $\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33} = \varepsilon_{12} = \varepsilon_{13} = 0$) $\rightarrow$ $\sigma_{12}^{I} = 2C_{1223}\varepsilon_{23}$

– Case II ($\varepsilon_{12} \neq 0$, $\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33} = \varepsilon_{13} = \varepsilon_{23} = 0$) $\rightarrow$ $\sigma_{23}^{II} = 2C_{2312}\varepsilon_{12}$

By isotropic condition, $\sigma_{12}^{I} = -\sigma_{23}^{II}$, for $\varepsilon_{12} = \varepsilon_{23}$.

$$2(C_{1223} + C_{2312})\varepsilon_{12} = 0 \rightarrow 4C_{1223}\varepsilon_{12} = 0 \rightarrow C_{1223} = 0$$





- δ expressions

Rel.	Coefficient	δ relation	Value
N-N	$C_{1111}=C_{2222}=C_{3333}$	$\delta_{ip}\delta_{jp}\delta_{kp}\delta_{lp}$	A_1
N-O.N.	$C_{1122}=C_{2233}=C_{1133}$ $(C_{2211}=C_{3322}=C_{3311})$	$\delta_{ij}\delta_{kl}-\delta_{ip}\delta_{jp}\delta_{kp}\delta_{lp}$	A_2
S-S	$C_{1212}=C_{2323}=C_{1313}$ $(C_{2112}=C_{3223}=C_{3113})$ $(C_{1221}=C_{2332}=C_{1331})$ $(C_{2121}=C_{3232}=C_{3131})$	$\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}-2\delta_{ip}\delta_{jp}\delta_{kp}\delta_{lp}$	A_3

The expressions in parentheses represent dependent relations.

- Superposition

$$C_{ijkl}=A_1\delta_{ip}\delta_{jp}\delta_{kp}\delta_{lp}+A_2(\delta_{ij}\delta_{kl}-\delta_{ip}\delta_{jp}\delta_{kp}\delta_{lp})+A_3(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}-2\delta_{ip}\delta_{jp}\delta_{kp}\delta_{lp})$$

$$C_{ijkl}=A_2\delta_{ij}\delta_{kl}+A_3(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})+(A_1-A_2-2A_3)\delta_{ip}\delta_{jp}\delta_{kp}\delta_{lp}$$

- Rotation of Stress and Strain

$$\sigma'_{ij}=\beta_{im}\sigma_{mn}\beta_{jn} \quad , \quad \varepsilon_{kl}=\beta_{pk}\varepsilon'_{pq}\beta_{ql}$$

$$\begin{aligned}\sigma'_{ij}&=\beta_{im}C_{mnkl}\beta_{jn}\beta_{pk}\varepsilon'_{pq}\beta_{ql}=\beta_{im}\beta_{jn}C_{mnkl}\beta_{pk}\beta_{ql}\varepsilon'_{pq}\\&=A_2\beta_{im}\beta_{jn}\delta_{mn}\delta_{kl}\beta_{pk}\beta_{ql}\varepsilon'_{pq}+A_3\beta_{im}\beta_{jn}(\delta_{mk}\delta_{nl}+\delta_{ml}\delta_{nk})\beta_{pk}\beta_{ql}\varepsilon'_{pq}+\\&\quad(A_1-A_2-2A_3)\beta_{im}\beta_{jn}\delta_{mr}\delta_{nr}\delta_{kr}\delta_{lr}\beta_{pk}\beta_{ql}\varepsilon'_{pq}\\&=A_2\beta_{im}\beta_{jm}\beta_{pk}\beta_{qk}\varepsilon'_{pq}+A_3[\beta_{im}\beta_{jn}\beta_{pm}\beta_{qn}+\beta_{im}\beta_{jn}\beta_{pn}\beta_{qm}]\varepsilon'_{pq}+\\&\quad(A_1-A_2-2A_3)\beta_{ir}\beta_{jr}\beta_{pr}\beta_{qr}\varepsilon'_{pq}\\&=A_2\delta_{ij}\delta_{pq}\varepsilon'_{pq}+A_3[\delta_{ip}\delta_{jq}+\delta_{iq}\delta_{jp}]\varepsilon'_{pq}+(A_1-A_2-2A_3)\beta_{ir}\beta_{jr}\beta_{pr}\beta_{qr}\varepsilon'_{pq}\\&=A_2\delta_{ij}\delta_{kl}\varepsilon'_{kl}+A_3[\delta_{il}\delta_{jl}+\delta_{il}\delta_{jk}]\varepsilon'_{kl}+(A_1-A_2-2A_3)\beta_{ip}\beta_{jp}\beta_{kp}\beta_{lp}\varepsilon'_{kl}\\&=A_2\delta_{ij}\delta_{kl}\varepsilon'_{kl}+A_3[\delta_{il}\delta_{jl}+\delta_{il}\delta_{jk}]\varepsilon'_{kl}+(A_1-A_2-2A_3)\delta_{ip}\delta_{jp}\delta_{kp}\delta_{lp}\varepsilon'_{kl}\\&=C'_{ijkl}\varepsilon'_{kl}\end{aligned}$$

For isotropy, $C_{ijkl}=C'_{ijkl}$

$$(A_1-A_2-2A_3)\beta_{ip}\beta_{jp}\beta_{kp}\beta_{lp}=(A_1-A_2-2A_3)\delta_{ip}\delta_{jp}\delta_{kp}\delta_{lp}$$

$$A_1-A_2-2A_3=0\rightarrow A_1=A_2+2A_3$$

- Final Relation: two independent coefficient

$$C_{ijkl}=A_2\delta_{ij}\delta_{kl}+A_3(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})=\lambda\delta_{ij}\delta_{kl}+\mu(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})$$

- Lame Elastic Constant, λ and μ

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \quad , \quad \mu = \frac{E}{2(1+\nu)}$$

- Detailed Expressions

$$\begin{aligned}\sigma_{ij} &= (\lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}))\varepsilon_{kl} \\ &= \lambda\varepsilon_{kk}\delta_{ij} + \mu\varepsilon_{ij} + \mu\varepsilon_{ji} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}\end{aligned}$$

$$\begin{aligned}\sigma_{11} &= (\lambda+2\mu)\varepsilon_{11} + \lambda\varepsilon_{22} + \lambda\varepsilon_{33} \\ \sigma_{22} &= \lambda\varepsilon_{11} + (\lambda+2\mu)\varepsilon_{22} + \lambda\varepsilon_{33} \\ \sigma_{33} &= \lambda\varepsilon_{11} + \lambda\varepsilon_{22} + (\lambda+2\mu)\varepsilon_{33} \\ \sigma_{12} &= \sigma_{21} = 2\mu\varepsilon_{12} = \mu\gamma_{12} \\ \sigma_{13} &= \sigma_{31} = 2\mu\varepsilon_{13} = \mu\gamma_{13} \\ \sigma_{23} &= \sigma_{32} = 2\mu\varepsilon_{23} = \mu\gamma_{23}\end{aligned}$$

- Equilibrium equation in terms of Lame constant

$$\sigma_{ij,j} + b_i = \rho\frac{\partial^2 u_i}{\partial t^2}$$

$$\lambda\frac{\partial}{\partial x_j}\frac{\partial u_k}{\partial x_k}\delta_{ij} + \mu\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + b_i = \rho\frac{\partial^2 u_i}{\partial t^2}$$

$$\lambda\frac{\partial}{\partial x_i}\frac{\partial u_k}{\partial x_k} + \mu\frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial x_j} + \mu\frac{\partial}{\partial x_i}\frac{\partial u_j}{\partial x_j} + b_i = \rho\frac{\partial^2 u_i}{\partial t^2}$$

$$(\lambda+\mu)\frac{\partial}{\partial x_i}\frac{\partial u_k}{\partial x_k} + \mu\frac{\partial^2 u_i}{\partial x_j \partial x_j} + b_i = \rho\frac{\partial^2 u_i}{\partial t^2}$$

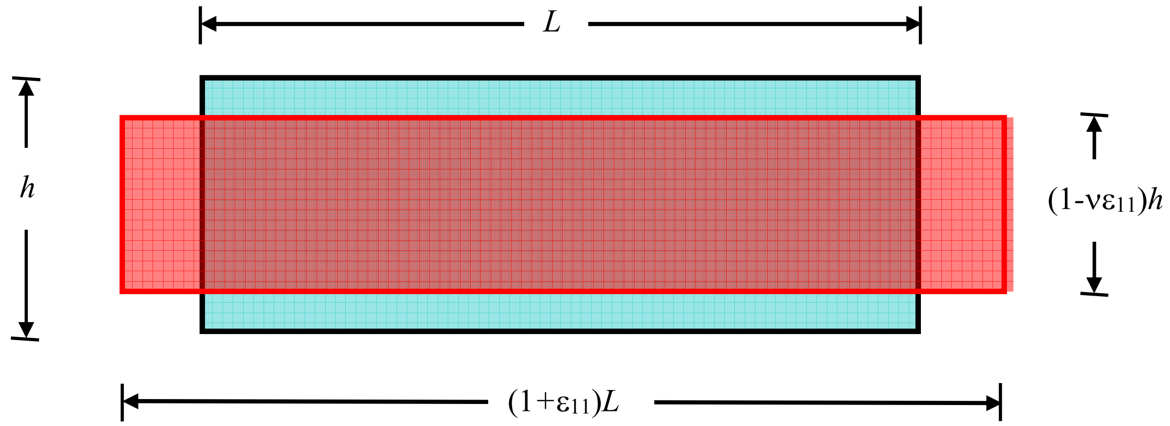
$$(\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}) + \mu\nabla^2\mathbf{u} + \mathbf{b} = \rho\frac{\partial^2\mathbf{u}}{\partial t^2}$$

or

$$(\lambda+\mu)\frac{\partial}{\partial x_i}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right) + \mu\left(\frac{\partial^2 u_i}{\partial x_1^2} + \frac{\partial^2 u_i}{\partial x_2^2} + \frac{\partial^2 u_i}{\partial x_3^2}\right) + b_i = \rho\frac{\partial^2 u_i}{\partial t^2}$$

4.3. Physical Meanings of Material Properties

- Young's Modulus, Poisson's Ratio & Lame Constant

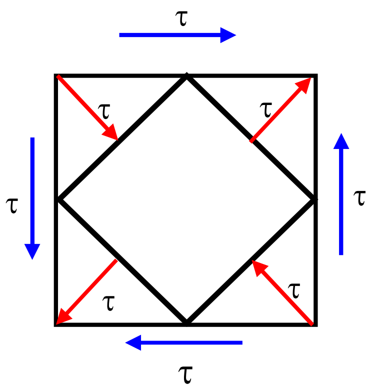


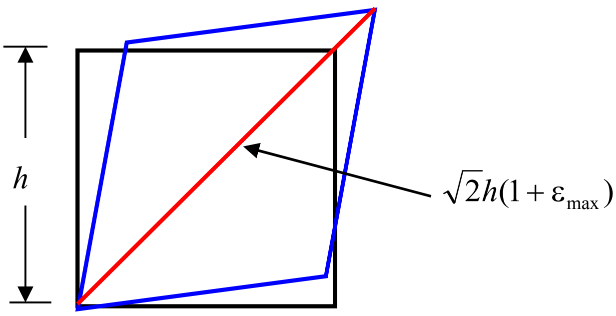
$$\varepsilon_{11} = \frac{\sigma_{11}}{E} - \nu\left(\frac{\sigma_{22}}{E} + \frac{\sigma_{33}}{E}\right)$$
$$\varepsilon_{22} = \frac{\sigma_{22}}{E} - \nu\left(\frac{\sigma_{11}}{E} + \frac{\sigma_{33}}{E}\right)$$
$$\varepsilon_{33} = \frac{\sigma_{33}}{E} - \nu\left(\frac{\sigma_{11}}{E} + \frac{\sigma_{22}}{E}\right)$$

$$\sigma_{11} = \frac{E}{(1+\nu)(1-2\nu)}((1-\nu)\varepsilon_{11} + \nu(\varepsilon_{22} + \varepsilon_{33}))$$
$$\sigma_{22} = \frac{E}{(1+\nu)(1-2\nu)}((1-\nu)\varepsilon_{22} + \nu(\varepsilon_{11} + \varepsilon_{33}))$$
$$\sigma_{33} = \frac{E}{(1+\nu)(1-2\nu)}((1-\nu)\varepsilon_{33} + \nu(\varepsilon_{11} + \varepsilon_{22}))$$

$$A_1 = \frac{(1-\nu)E}{(1+\nu)(1-2\nu)} \quad , \quad A_2 = \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \quad , \quad A_3 = \mu = \frac{A_1 - A_2}{2} = \frac{E}{2(1+\nu)} = G$$

- Physical Derivation of Shear Modulus





$$\varepsilon_{max} = \frac{\tau}{E} + \nu\frac{\tau}{E} = \frac{\tau}{E}(1+\nu)$$

$$[\sqrt{2}h(1+\varepsilon_{max})]^2 = h^2 + h^2 - 2h^2\cos(\frac{\pi}{2}+\gamma)$$

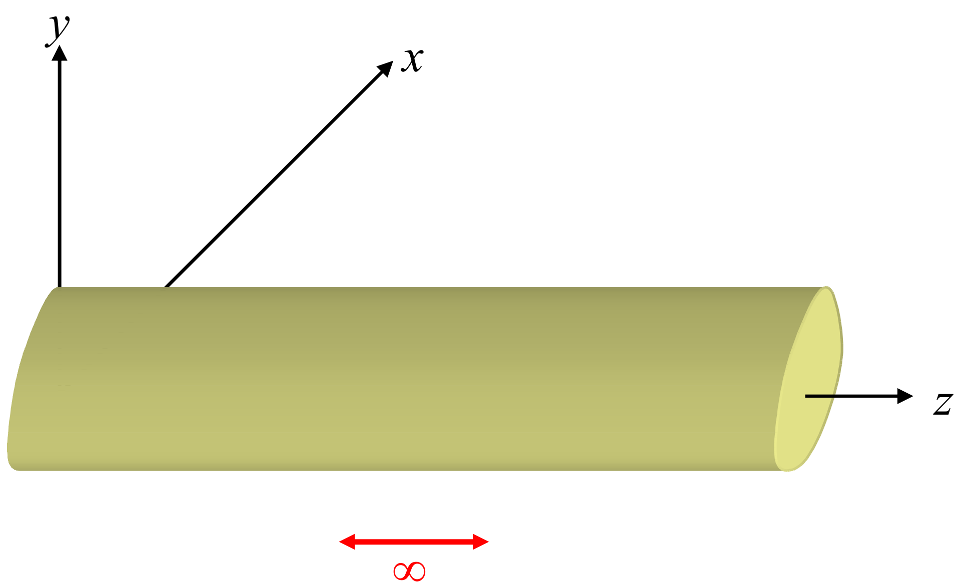
$$(1+\varepsilon_{max})^2 = 1+\sin(\gamma) \rightarrow 1+2\varepsilon_{max} \approx 1+\sin(\gamma) \approx 1+\gamma$$

$$\varepsilon_{max} = \frac{\gamma}{2} = \frac{\tau}{E}(1+\nu) \rightarrow \tau = \frac{E}{2(1+\nu)}\gamma \rightarrow \tau = G\gamma$$

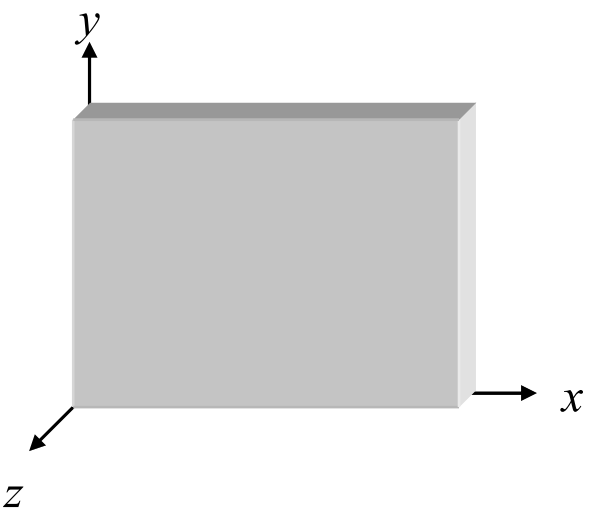
$$G = \frac{E}{2(1+\nu)}$$

Chapter 5

Plane Problems in Cartesian Coordinate System



5.1. Plane Stress



- Assumption
  - The thickness of the plate is small compared to the other dimensions of the plate.
  - No z- directional traction is applied on the surface.
  - The variation of stress through the thickness is neglected.

- Traction boundary condition

$$T_i = \sigma_{ij} n_j$$

  - On $\pm x - y$ plane: $\mathbf{n} = (0,0,\pm 1)$

$$- \quad T_x = \pm\sigma_{xz} = 0 \quad , \quad T_y = \pm\sigma_{yz} = 0 \quad , \quad T_z = \pm\sigma_{zz} = 0$$

  - By the third assumption

$$\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0 \quad \text{in } V$$

- Equilibrium Equation

$$\begin{aligned} &\frac{\partial\sigma_{11}}{\partial x_1} + \frac{\partial\sigma_{12}}{\partial x_2} + \cancel{\frac{\partial\sigma_{13}}{\partial x_3}} + b_1 = 0 \\ &\frac{\partial\sigma_{21}}{\partial x_1} + \frac{\partial\sigma_{22}}{\partial x_2} + \cancel{\frac{\partial\sigma_{23}}{\partial x_3}} + b_2 = 0 \\ &\cancel{\frac{\partial\sigma_{31}}{\partial x_1}} + \cancel{\frac{\partial\sigma_{32}}{\partial x_2}} + \cancel{\frac{\partial\sigma_{33}}{\partial x_3}} + b_3 = 0 \end{aligned} \quad \rightarrow \quad \begin{aligned} &\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} + b_x = 0 \\ &\frac{\partial\sigma_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} + b_y = 0 \\ &0 = 0 \end{aligned}$$

- Strain-Displacement relations

$$u_x = u_x(x,y) \ , \quad u_y = u_y(x,y) \ , \quad u_z : \text{no effect on solution}$$

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} \ , \quad \varepsilon_{yy} = \frac{\partial u_y}{\partial y} \ , \quad \varepsilon_{xy} = \frac{1}{2}(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x})$$

- Constitutive law

$$\left.\begin{aligned} \varepsilon_{xx} &= \frac{\sigma_{xx}}{E} - \nu\frac{\sigma_{yy}}{E} \\ \varepsilon_{yy} &= \frac{\sigma_{yy}}{E} - \nu\frac{\sigma_{xx}}{E} \\ \varepsilon_{zz} &= -\nu(\frac{\sigma_{xx}}{E} + \frac{\sigma_{yy}}{E}) = -\frac{\nu}{1-\nu}(\varepsilon_{xx} + \varepsilon_{yy}) \\ \varepsilon_{xy} &= \frac{\sigma_{xy}}{2G} \end{aligned}\right\} \rightarrow \left\{\begin{aligned} \sigma_{xx} &= \frac{E}{1-v^2}(\varepsilon_{xx} + v\varepsilon_{yy}) \\ \sigma_{yy} &= \frac{E}{1-v^2}(\varepsilon_{yy} + v\varepsilon_{xx}) \\ \sigma_{xy} &= 2G\varepsilon_{xy} \end{aligned}\right.$$

- Displacement method
  - Stress in terms of displacement

$$\sigma_{xx} = \frac{E}{1-v^2}(\frac{\partial u_x}{\partial x} + v\frac{\partial u_y}{\partial y})$$

$$\sigma_{yy} = \frac{E}{1-v^2}(\frac{\partial u_y}{\partial y} + v\frac{\partial u_x}{\partial x})$$

$$\sigma_{xy} = G(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x})$$

  - Equilibrium equation in terms of displacement

$$2(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2}) - (1+v)\frac{\partial}{\partial y}(\frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x}) + 2(1-v^2)\frac{b_x}{E} = 0$$

$$2(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2}) - (1+v)\frac{\partial}{\partial x}(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}) + 2(1-v^2)\frac{b_y}{E} = 0$$

  - Compatibility equations are automatically satisfied.

- Force method
  - Body Force: Potential Field

$$b_x = -\frac{\partial \psi}{\partial x} \ , \ b_y = -\frac{\partial \psi}{\partial y}$$

  - Airy's Stress function

$$\sigma_{xx} - \psi = \frac{\partial^2 \phi}{\partial y^2} \ , \ \sigma_{yy} - \psi = \frac{\partial^2 \phi}{\partial x^2} \ , \ \sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$$

  –

– Equilibrium Equation: satisfied identically

$$\frac{\partial}{\partial x}\left(\frac{\partial^2\phi}{\partial y^2}+\psi\right)-\frac{\partial}{\partial y}\left(\frac{\partial^2\phi}{\partial x\partial y}\right)-\frac{\partial\psi}{\partial x}=0 \rightarrow 0=0$$

$$-\frac{\partial}{\partial x}\left(\frac{\partial^2\phi}{\partial x\partial y}\right)+\frac{\partial}{\partial y}\left(\frac{\partial^2\phi}{\partial x^2}+\psi\right)-\frac{\partial\psi}{\partial y}=0 \rightarrow 0=0$$

– Strains in terms of stress function

$$\varepsilon_{xx}=\frac{1}{E}\left(\frac{\partial^2\phi}{\partial y^2}+\psi\right)-\frac{\nu}{E}\left(\frac{\partial^2\phi}{\partial x^2}+\psi\right)$$

$$\varepsilon_{yy}=\frac{1}{E}\left(\frac{\partial^2\phi}{\partial x^2}+\psi\right)-\frac{\nu}{E}\left(\frac{\partial^2\phi}{\partial y^2}+\psi\right)$$

$$\gamma_{xy}=\frac{\sigma_{xy}}{G}=-\frac{2(1+\nu)}{E}\frac{\partial^2\phi}{\partial x\partial y}$$

– Compatibility equation

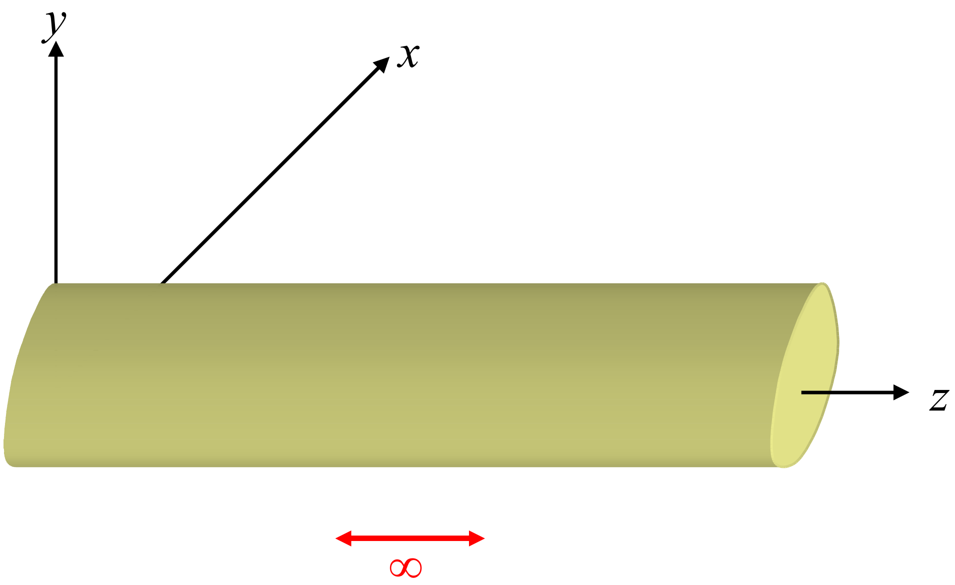
$$\frac{\partial^2\gamma_{12}}{\partial x_1\partial x_2}=\frac{\partial^2\varepsilon_{11}}{\partial x_2^2}+\frac{\partial^2\varepsilon_{22}}{\partial x_1^2}$$

$$2(1+\nu)\frac{\partial^4\phi}{\partial x^2\partial y^2}+\left(\frac{\partial^4\phi}{\partial y^4}+\frac{\partial^2\psi}{\partial y^2}\right)-\nu\left(\frac{\partial^4\phi}{\partial y^2\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)+\left(\frac{\partial^4\phi}{\partial x^4}+\frac{\partial^2\psi}{\partial x^2}\right)-\nu\left(\frac{\partial^4\phi}{\partial x^2\partial y^2}+\frac{\partial^2\psi}{\partial x^2}\right)=0$$

$$\frac{\partial^4\phi}{\partial x^4}+2\frac{\partial^4\phi}{\partial x^2\partial y^2}+\frac{\partial^4\phi}{\partial y^4}+(1-\nu)\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)=0$$

$$\nabla^4\phi=-(1-\nu)\nabla^2\psi$$

5.2. Plane Strain



- Assumption
  - Infinite Structure in z-direction with an identical section.
  - Identical traction in x, y-direction
  - No z- directional traction is applied on the surface.

- Strain-Displacement Relation

$$u_x = u_x(x,y) \;,\quad u_y = u_y(x,y) \;,\quad u_z = 0$$

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} \;,\quad \varepsilon_{yy} = \frac{\partial u_y}{\partial y} \;,\quad \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$$

$$\varepsilon_{zz} = \frac{\partial u_z}{\partial x} = 0 \,,\; \varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) = 0 \,,\; \varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) = 0$$

- Constitutive law

$$\varepsilon_{zz} = \frac{\sigma_{zz}}{E} - \nu\left(\frac{\sigma_{xx}}{E} + \frac{\sigma_{yy}}{E}\right) = 0 \;\rightarrow\; \sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})$$

$$\varepsilon_{xx} = (1-\nu^2)\frac{\sigma_{xx}}{E} - (\nu + \nu^2)\frac{\sigma_{yy}}{E}$$

$$\varepsilon_{yy} = (1-\nu^2)\frac{\sigma_{yy}}{E} - (\nu + \nu^2)\frac{\sigma_{xx}}{E}$$

$$\gamma_{xy} = \frac{2(1+\nu)}{E}\sigma_{xy}$$

5.3 Polynomial Stress Functions

- Compatability Condition w/o body forces

$$\nabla^4\phi = \frac{\partial^4\phi}{\partial x^4} + 2\frac{\partial^4\phi}{\partial x^2 \partial y^2} + \frac{\partial^4\phi}{\partial y^4} = 0$$

- Stress Components

$$\sigma_{xx} = \frac{\partial^2\phi}{\partial y^2} \;,\; \sigma_{yy} = \frac{\partial^2\phi}{\partial x^2} \;,\; \sigma_{xy} = -\frac{\partial^2\phi}{\partial x \partial y}$$

Do we have any robust strategy to solve the compatibility equation? Unfortunately, the answer is “No”! The basic strategy to find the solutions of the C.E. is the trial and error method.

- Trial and Error Solution Procedures

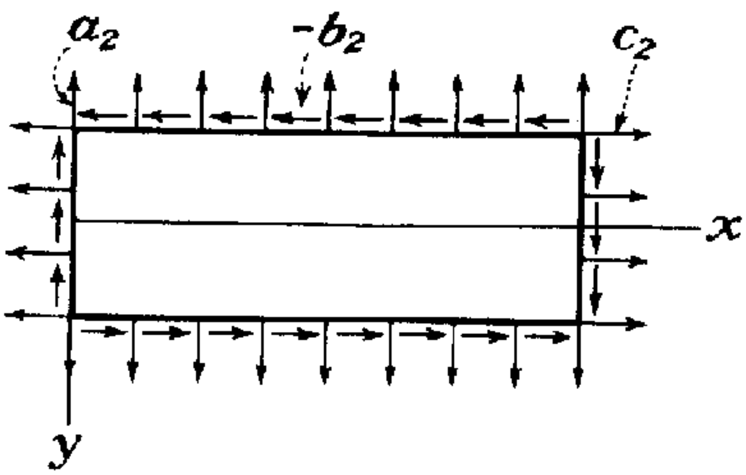
  1. Predefine several Airy’s stress functions satisfying the compatibility equation.
  2. Combine the predefined functions to satisfy given traction boundary conditions.
  3. Calculate strain and displacement.
  4. Check the displacement boundary conditions.
  5. If the DBCs are satisfied, you have the solution. If not, try another combination!

- Second order

$$\phi_2 = \frac{a_2}{2}x^2 + b_2 xy + \frac{c_2}{2}y^2$$

$$\sigma_{xx} = \frac{\partial^2\phi}{\partial y^2} = c_2 \;,\; \sigma_{yy} = \frac{\partial^2\phi}{\partial x^2} = a_2 \;,\; \sigma_{xy} = -\frac{\partial^2\phi}{\partial x \partial y} = -b_2$$

  – Constant stress status



- Third Order

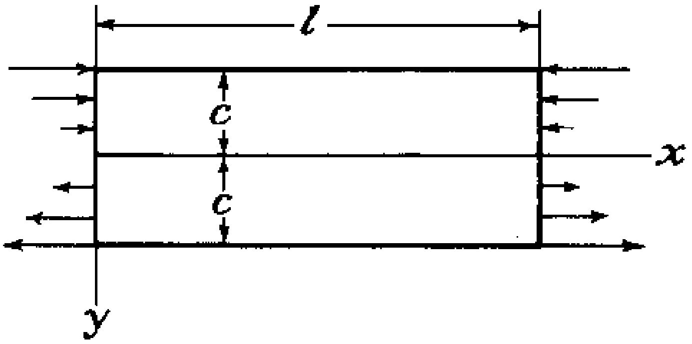
$$\phi_3 = \frac{a_3}{3\cdot 2}x^3 + \frac{b_3}{2}x^2 y + \frac{c_3}{2}xy^2 + \frac{d_3}{3\cdot 2}y^3$$

$$\sigma_{xx} = \frac{\partial^2\phi}{\partial y^2} = c_3 x + d_3 y$$

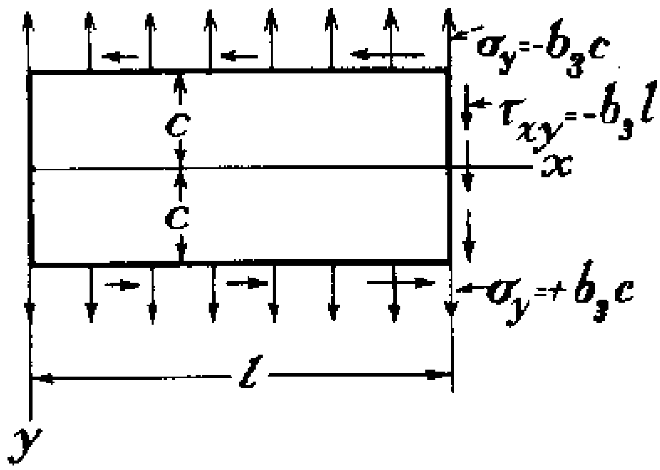
$$\sigma_{yy} = \frac{\partial^2\phi}{\partial x^2} = a_3 x + b_3 y$$

$$\sigma_{xy} = -\frac{\partial^2\phi}{\partial x \partial y} = -b_3 x - c_3 y$$

- Linear varying stress status in x-y direction
- $d_3 \neq 0$ or $a_3 \neq 0$: pure bending cases



- $b_3 \neq 0$: constant normal stress and linearly varying shear stress

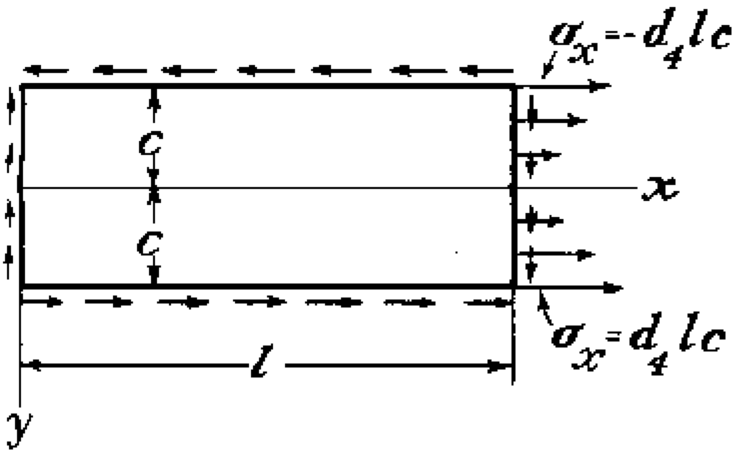


- Fourth order

$$\phi_4 = \frac{a_4}{4\cdot 3}x^4 + \frac{b_4}{3\cdot 2}x^3y + \frac{c_4}{2}x^2y^2 + \frac{d_4}{3\cdot 2}xy^3 + \frac{e_4}{4\cdot 3}y^4$$
$$e_4 = -(2c_4 + a_4)$$
$$\sigma_{xx} = \frac{\partial^2\phi}{\partial y^2} = c_4x^2 + d_4xy - (2c_4 + a_4)y^2$$
$$\sigma_{yy} = \frac{\partial^2\phi}{\partial x^2} = a_4x^2 + b_4xy + c_4y^2$$
$$\sigma_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} = -\frac{b_4}{2}x^2 - 2c_4xy - \frac{d_4}{2}y^2$$

– $d_4 \neq 0$:

$$\sigma_{xx} = d_4xy \ , \ \sigma_{yy} = 0 \ , \ \sigma_{xy} = -\frac{d_4}{2}y^2$$



$$\text{Moment by S. stress} = \frac{d_4c^2l}{2}2c - \frac{1}{3}\frac{d_4c^2}{2}2cl = \frac{2}{3}d_4c^3l$$

$$\text{Moment by N. stress} = \frac{d_4c^2l}{2}\frac{2}{3}2c = \frac{2}{3}d_4c^3l$$

- Fifth order

$$\phi_5 = \frac{a_5}{5\cdot 4}x^5 + \frac{b_5}{4\cdot 3}x^4 y + \frac{c_5}{3\cdot 2}x^3 y^2 + \frac{d_5}{3\cdot 2}x^2 y^3 + \frac{e_5}{4\cdot 3}xy^4 + \frac{f_5}{5\cdot 4}y^5$$

$$e_5 = -(2c_5 + 3a_5) \quad , \quad f_5 = -\frac{1}{3}(b_5 + 2d_5)$$

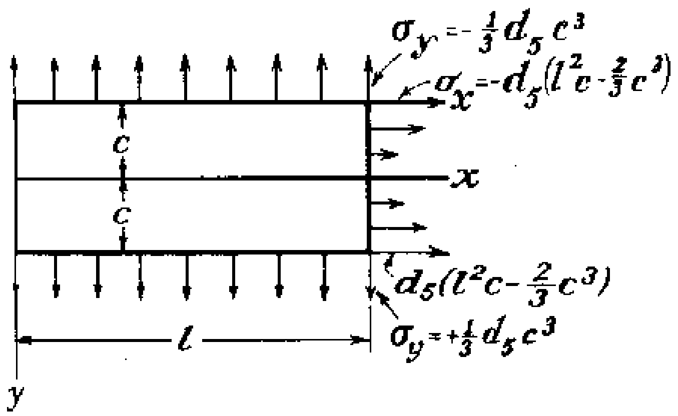
$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} = \frac{c_5 x^3}{3} + d_5 x^2 y - (2c_5 + 3a_5)xy^2 - \frac{1}{3}(b_5 + 2d_5)y^3$$

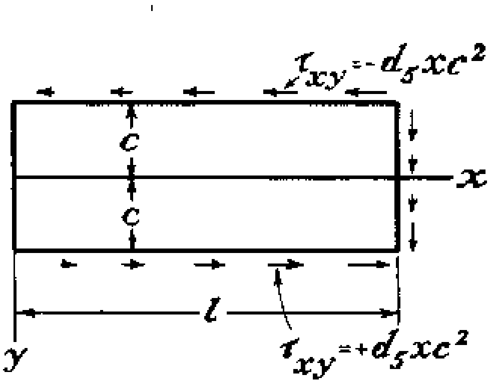
$$\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} = a_5 x^3 + b_5 x^2 y + c_5 xy^2 + \frac{d_5}{3}y^3$$

$$\sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = -\frac{b_5}{3}x^3 - c_5 x^2 y - d_5 xy^2 + \frac{1}{3}(2c_5 + 3a_5)y^3$$

– $d_5 \neq 0$:

$$\sigma_{xx} = d_5(x^2 y - \frac{2}{3}y^3) \ , \ \sigma_{yy} = \frac{d_5}{3}y^3 \ \ , \ \sigma_{xy} = -d_5 xy^2$$





- Final Solution

$$\phi = \sum_i \phi_i$$

  – Coefficient of each polynomial should be selected so that the traction boundary conditions are satisifed.

$$T_i = \sigma_{ij} n_j$$

  – The tractions on the surface where displacement is specified are determined once the S.F. is obtained.

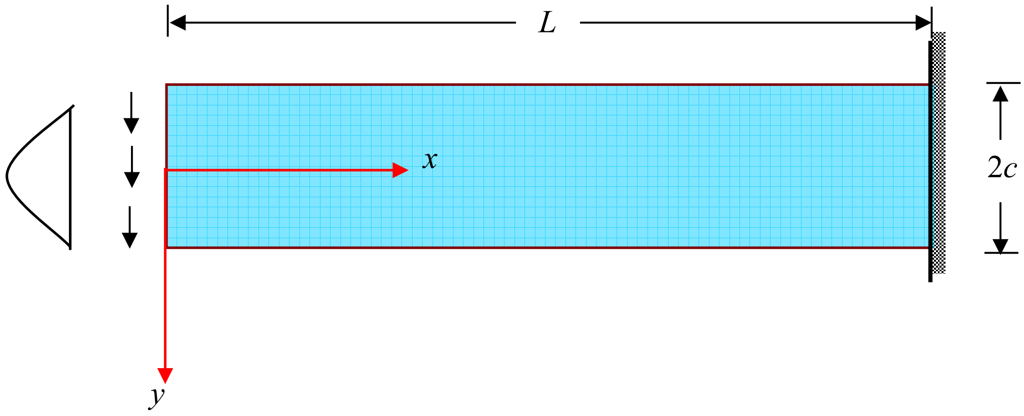
  – The displacement field is obtained by integrating the strain field.

- Saint-Venant's Principle

In the elsatostatics, if the boundary tractions ($\overline{\mathbf{T}}$) on a part S_1 of the boundary S are replaced by a statically equilvalent traction distribution ($\overline{\mathbf{T}}'$), the effects on the stress distribution in the body are negligible at points whose distance from S1 is large compared to the maximum distance between points of S_1.

$$\int_{S_1} \overline{\mathbf{T}}' dS = \int_{S_1} \overline{\mathbf{T}} dS, \quad \int_{S_1} \overline{\mathbf{T}}' \times \mathbf{r} dS = \int_{S_1} \overline{\mathbf{T}} \times \mathbf{r} dS$$

5.4. Cantilever Beam



- Traction Boundary Conditions
  - on $y = \pm c$
    $T_x = \sigma_{xi} n_i = \pm \sigma_{xy} = 0$, $T_y = \sigma_{yi} n_i = \pm \sigma_{yy} = 0$,
  - on $x = 0$
    $T_x = \sigma_{xi} n_i = -\sigma_{xx} = 0$, $T_y = \sigma_{yi} n_i = -\sigma_{yx}$

- Assumption
  - Stress distributions can be determined by the beam theory.

$$\sigma_{xx} = d_4 xy \quad , \quad \sigma_{yy} = 0 \quad , \quad \sigma_{xy} = -b_2 - \frac{d_4 y^2}{2}$$

- Boundary condition on $y = \pm c$

$$(\sigma_{xy})_{y=\pm c} = -b_2 - \frac{d_4 c^2}{2} = 0 \rightarrow d_4 = -2\frac{b_2}{c^2}$$

- Statical equivalence

$$\int_{-c}^{c} (T_y)_{x=0} t dy = -\int_{-c}^{c} \sigma_{xy} t dy = \int_{-c}^{c} (b_2 - \frac{b_2}{c^2} y^2) t dy = P \rightarrow b_2 = \frac{3}{4}\frac{P}{ct}$$

- Stress

$$\sigma_{xx} = -\frac{3}{2}\frac{P}{c^3 t} xy = -\frac{Px}{I} y = -\frac{M}{I} y$$
$$\sigma_{yy} = 0$$
$$\sigma_{xy} = -\frac{P}{2I}(c^2 - y^2)$$

- Displacement

  – Strain

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} = \frac{\sigma_{xx}}{E} - \nu\frac{\sigma_{yy}}{E} = -\frac{P}{EI}xy$$
$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y} = -\nu\frac{\sigma_{xx}}{E} + \frac{\sigma_{yy}}{E} = \frac{\nu P}{EI}xy$$
$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = \frac{\sigma_{xy}}{G} = -\frac{P}{2IG}(c^2 - y^2)$$

  – Integration

$$u_x = -\frac{P}{2EI}x^2 y + f(y) \ , \ \ u_y = \frac{\nu P}{2EI}xy^2 + f_1(x)$$

  – By shear strain

$$\frac{\partial}{\partial y}(-\frac{P}{2EI}x^2 y + f(y)) + \frac{\partial}{\partial x}(\frac{\nu P}{2EI}xy^2 + f_1(x)) = -\frac{P}{2IG}(c^2 - y^2)$$
$$(-\frac{P}{2EI}x^2 + \frac{df}{dy}) + (\frac{\nu P}{2EI}y^2 + \frac{df_1}{dx}) = -\frac{P}{2IG}(c^2 - y^2)$$
$$(-\frac{P}{2EI}x^2 + \frac{df_1}{dx}) + (\frac{\nu P}{2EI}y^2 + \frac{df}{dy} - \frac{P}{2IG}y^2) = -\frac{Pc^2}{2IG}$$
$$F_x(x) + F_y(y) = -\frac{Pc^2}{2IG}$$

  – F_x and F_y should be constant.

$$F_x = -\frac{P}{2EI}x^2 + \frac{df_1}{dx} = d \ , \ \ F_y = \frac{\nu P}{2EI}y^2 + \frac{df}{dy} - \frac{P}{2IG}y^2 = e$$

  – Integration of F_x and F_y

$$f_1 = \frac{P}{6EI}x^3 + dx + h \ , \ \ f = -\frac{\nu P}{6EI}y^3 + \frac{P}{6IG}y^3 + ey + g$$

  – Displacement components

$$u_x = -\frac{P}{2EI}x^2 y - \frac{\nu P}{6EI}y^3 + \frac{P}{6IG}y^3 + ey + g$$
$$u_y = \frac{\nu P}{2EI}xy^2 + \frac{P}{6EI}x^3 + dx + h$$

- Displacement Boundary Conditions
  - on $x = l$

    $u_x = 0$, $u_y = 0$

    $$u_x = -\frac{P}{2EI} l^2 y - \frac{\nu P}{6EI} y^3 + \frac{P}{6IG} y^3 + ey + g = (-\frac{P}{2EI} l^2 + e) y + (-\frac{\nu P}{6EI} + \frac{P}{6IG}) y^3 + g = 0 \rightarrow g = 0, e = \frac{Pl^2}{2EI}, \frac{\nu P}{6EI} = \frac{P}{6IG}$$

  - however, the third condition is impossible.

    $$\frac{\nu P}{6EI} = \frac{P}{6IG} \rightarrow E = \nu G \rightarrow E = \frac{\nu E}{2(1+\nu)} \rightarrow \nu = -2$$

  - y-Displacement on $x = l$

    $$u_y = \frac{\nu P}{2EI} l y^2 + \frac{P}{6EI} l^3 + dl + h = 0 \rightarrow \frac{\nu P}{2EI} l = 0 \text{ and } \frac{P}{6EI} l^3 + dl + h = 0$$

    $\nu = 0$ or $P = 0$

- We end up wirh a useless solution and the displacement BC cannot be satisfied !!!

  Why ???

- Question on DBC

  Since we have 4 integration constants and only one condition ( ) for them, three conditions at some points (not on a surface) are required to determine the integration constant. Bounda-ry conditions specified on a surface can be enforced exactly with a function on the surface, rather than with just a few constants. Generally, the integration constants for a partial differential equa-tion with boundary conditions defined on a surface should be a function defined on the surface. Therefore, it is impossible to enforce boundary conditions defined on a surface with a few constants. The assumptions on the stress field for this problem based on the beam solution cannot satisfy the fixed end condition on the whole right boundary, and thus in principle we have to guess another stress field, which is very difficult. Anyway, let's try to determine the integration constant with three displacement boundary conditions given at the center of the fixed end.

- DBC on the neutral axis at the support.
  - at $x = l$ and $y = 0$ $u_x = u_y = 0$

    $u_x(l,0) = g = 0$

    $u_y(l,0) = \frac{P}{6EI} l^3 + dl + h = 0 \rightarrow h = -\frac{P}{6EI} l^3 - dl$

  - at $x = l$ and $y = 0$ $\frac{\partial u_y}{\partial x} = 0$

    $\frac{\partial u_y}{\partial x} = \frac{P}{2EI} l^2 + d = 0 \rightarrow d = -\frac{P}{2EI} l^2 \rightarrow h = \frac{P}{3EI} l^2$

    $e = \frac{P}{2EI} l^2 - \frac{Pc^2}{2GI}$

    $u_x = -\frac{P}{2EI} x^2 y - \frac{\nu P}{6EI} y^3 + \frac{P}{6IG} y^3 + (\frac{Pl^2}{2EI} - \frac{Pc^2}{2GI}) y$

    $u_y = \frac{\nu P}{2EI} xy^2 + \frac{P}{6EI} x^3 - \frac{Pl^2}{2EI} x + \frac{Pl^3}{3EI}$

  - Verification of solution

    $\frac{\nu P}{2EI} lc^2 / \frac{Pl^3}{3EI} = \frac{3}{2} \frac{c^2}{l^2} \nu \approx 0.45\%$ for $\frac{c}{l} = 0.1$ and ν=0.3

    $u_x = -\frac{P}{2EI} x^2 y - \frac{\nu P}{6EI} y^3 + \frac{P}{6IG} y^3 + (\frac{Pl^2}{2EI} - \frac{Pc^2}{2GI}) y$

    $= \frac{P}{2EI}(-x^2 + l^2) y - \frac{\nu P}{6EI} y^3 + \frac{P}{6GI}(y^3 - 3c^2 y)$

    Beam solution $\varepsilon_{xx} = -\frac{Px}{EI} y \rightarrow u_x = -\frac{P}{2EI}(-x^2 + l^2) y$

    $\frac{Pc^3}{3GI} / \frac{Pl^2 c}{2EI} = \frac{4}{3}(1+\nu)(\frac{c}{l})^2 \approx 1.7\%$

    $\nu \frac{Pc^3}{6EI} / \frac{Pl^2 c}{2EI} = \frac{1}{3} \nu (\frac{c}{l})^2 \approx 0.1\%$

    for $\frac{c}{l} = 0.1$ and ν=0.3

  - at $x = l$ and $y = 0$ $\frac{\partial u_x}{\partial y} = 0$

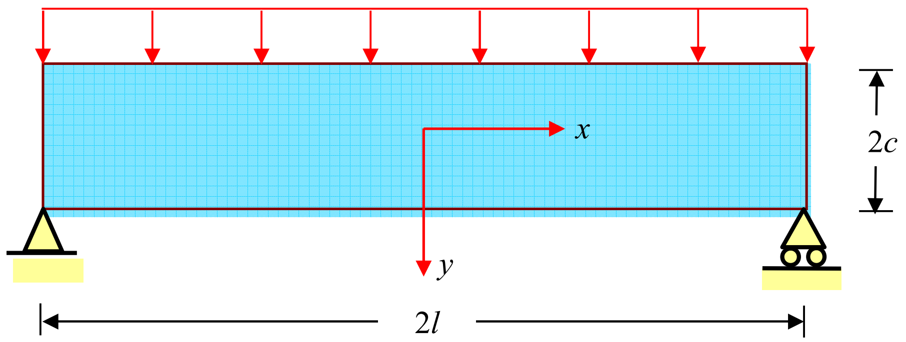
    $\frac{\partial u_x}{\partial y} = -\frac{Pl^2}{2EI} + e = 0 \rightarrow e = \frac{Pl^2}{2EI} \rightarrow d = -\frac{P}{2EI} l^2 - \frac{Pc^2}{2GI}$

    $h = \frac{P}{3EI} l^3 + \frac{Pc^2 l}{2GI}$

    $u_x = -\frac{P}{2EI}(x^2 - l)^2 y - \frac{\nu P}{6EI} y^3 + \frac{P}{6IG} y^3$

    $u_y = \frac{P}{6EI} x^3 - \frac{Pl^2}{2EI} x + \frac{Pl^3}{3EI} + \frac{\nu P}{2EI} xy^2 + \frac{Pc^2}{2GI}(l - x)$

5.5. Simple Beam



- Traction boundary conditions

i. on $y = -c$

$T_1 = \sigma_{xj} n_j = -\sigma_{xy} = 0 \rightarrow \sigma_{xy} = 0$,

$T_2 = \sigma_{yj} n_j = -\sigma_{yy} = q \rightarrow \sigma_{yy} = -q$

ii. on $y = c$

$T_1 = \sigma_{xj} n_j = \sigma_{xy} = 0$, $T_2 = \sigma_{yj} n_j = \sigma_{yy} = 0$

iii. on $x = -l$

$T_1 = \sigma_{xj} n_j = -\sigma_{xx} = 0$, $T_2 = \sigma_{yj} n_j = -\sigma_{yx} = 0$

iv. on $x = l$

$T_1 = \sigma_{xj} n_j = \sigma_{xx} = 0$, $T_2 = \sigma_{yj} n_j = \sigma_{yx} = 0$

- Symmetry conditions

i. σ_{22} should not vary along x on constant y.

ii. σ_{12} should be anti-symmetry with respect to $x \rightarrow$ should be odd function wrt x.

- Selection of functions

i. $n = 2$

$\sigma_{11} = c_2$, $\sigma_{22} = a_2$, $\sigma_{12} = -b_2$

From the SM condition ii, $b_2 = 0$

ii. $n = 3$

$\sigma_{11} = c_3 x + d_3 y$, $\sigma_{22} = a_3 x + b_3 y$, $\sigma_{12} = -b_3 x - c_3 y$

From the SM condition i, $a_3 = 0$

From the SM condition ii, $c_3 = 0$

iii. $n = 4$

$$\sigma_{11} = c_4x^2 + d_4xy - (2c_4 + a_4)y^2$$
$$\sigma_{22} = a_4x^2 + b_4xy + c_4y^2$$
$$\sigma_{12} = -\frac{b_4x^2}{2} - 2c_4xy - d_4y^2$$

From the SM condition 1, $a_4 = 0$, $b_4 = 0$
From the SM condition 2, $d_4 = 0$

iv. $n = 5$

$$\sigma_{11} = \frac{c_5x^3}{3} + d_5x^2y - (2c_5 + 3a_5)xy^2 - \frac{1}{3}(b_5 + 2d_5)y^3$$
$$\sigma_{22} = a_5x^3 + b_5x^2y + c_5xy^2 + \frac{d_5y^3}{3}$$
$$\sigma_{12} = -\frac{b_5x^3}{3} - c_5xy - d_5xy^2 + \frac{1}{3}(2c_5 + 3a_5)y^3$$

From the SM condition 1, $a_5 = 0$, $b_5 = 0$, $c_5 = 0$

- Final Expressions of Stress components

$$\sigma_{11} = c_2 + d_3y + c_4(x^2 - 2y^2) + d_5(x^2y - \frac{2}{3}y^3)$$
$$\sigma_{22} = a_2 + b_3y + c_4y^2 + \frac{d_5y^3}{3}$$
$$\sigma_{12} = -b_3x - 2c_4xy - d_5xy^2$$

- Apply the given boundary conditions
  - The boundary condition 1
  $$-b_3x + 2c_4cx - d_5xc^2 = 0 \rightarrow b_3 - 2c_4c + d_5c^2 = 0$$
  $$a_2 - b_3c + c_4c^2 - \frac{d_5c^3}{3} = -q$$
  - The boundary condition 2
  $$-b_3x - 2c_4cx - d_5xc^2 = 0 \rightarrow b_3 + 2c_4c + d_5c^2 = 0$$
  $$a_2 + b_3c + c_4c^2 + \frac{d_5c^3}{3} = 0$$
  - Solve the simultaneous equations
  $$a_2 = -\frac{q}{2},\ b_3 = \frac{3}{4}\frac{q}{c} = \frac{qc^2}{2I},\ c_4 = 0,\ d_5 = -\frac{3}{4}\frac{q}{c^3} = -\frac{q}{2I}$$

- Apply the static equilibrium conditions

  - The static equilibrium condition 1: $\int_{-c}^{c} \sigma_{11}\big|_{x=\pm l} dy = 0$

$$\int_{-c}^{c} (c_2 + d_3 y - \frac{q}{2I}(l^2 y - \frac{2}{3}y^3))dy = 2cc_2 = 0 \rightarrow c_2 = 0$$

  - The static equilibrium condition 2: $\int_{-c}^{c} \sigma_{11}\big|_{x=\pm l} ydy = 0$

$$\int_{-c}^{c} (d_3 y^2 - \frac{q}{2I}(l^2 y^2 - \frac{2}{3}y^4))dy = \frac{2d_3 c^3}{3} - \frac{q}{I}(\frac{l^2 c^3}{3} - \frac{2}{15}c^5) = 0$$

$$d_3 = \frac{q}{2I}(l^2 - \frac{2}{5}c^2)$$

- The final expression for stress

$$\sigma_{11} = \frac{q}{2I}(l^2 - x^2)y + \frac{q}{2I}(\frac{2}{3}y^3 - \frac{2}{5}c^2 y)$$

$$\sigma_{22} = -\frac{q}{2I}(\frac{2}{3}c^3 - c^2 y + \frac{y^3}{3})$$

$$\sigma_{12} = -\frac{q}{2I}(c^2 - y^2)x$$

- The static equilibrium conditions:

$$\int_{-c}^{c} \sigma_{12}\big|_{x=\pm l} dy = \mp ql \qquad \mp \int_{-c}^{c} \frac{q}{2I}(c^2 - y^2)ldy = \mp ql$$

5.6. Series Solution

$$\nabla^4\phi = \frac{\partial^4\phi}{\partial x^4} + 2\frac{\partial^4\phi}{\partial x^2\partial y^2} + \frac{\partial^4\phi}{\partial y^4} = 0$$

$$\phi_m = (X_1 \sin\frac{m\pi x}{l} + X_2 \cos\frac{m\pi x}{l}) f(y) \quad , \quad \phi = \sum_{m=0}^{\infty}\phi_m$$

- In case either sine function or cosine function is used,

$$\phi_m = \sin\frac{m\pi x}{l} f(y) = \sin\alpha x f(y) \quad , \quad \phi = \sum_{m=0}^{\infty}\phi_m$$

- Compatibility equation for one sine function.

$$\alpha^4 f(y) - 2\alpha^2 f''(y) + f''''(y) = 0$$

- General solution

$$f(y) = C_1\cosh\alpha y + C_2\sinh\alpha y + C_3 y\cosh\alpha y + C_4 y\sinh\alpha y$$

- Stress components

$$\sigma_{xx} = \sin\alpha x[\alpha^2 C_1\cosh\alpha y + \alpha^2 C_2\sinh\alpha y + \alpha C_3(2\sinh\alpha y + \alpha y\cosh\alpha y) + \alpha C_4(2\cosh\alpha y + \alpha y\sinh\alpha y)]$$

$$\sigma_{yy} = -\alpha^2\sin\alpha x[C_1\cosh\alpha y + C_2\sinh\alpha y + C_3 y\cosh\alpha y + C_4 y\sinh\alpha y]$$

$$\sigma_{xy} = -\alpha\cos\alpha x[\alpha C_1\sinh\alpha y + \alpha C_2\cosh\alpha y + C_3(\alpha\cosh y + \alpha y\sinh\alpha y) + C_4(\sinh\alpha y + \alpha y\cosh\alpha y)]$$

- Traction Boundary conditions
  - on $y = c$

    $\sigma_{xy} = 0 \quad , \quad \sigma_{yy} = -B_m\sin\alpha x$
  - on $y = -c$

    $\sigma_{xy} = 0 \quad , \quad \sigma_{yy} = -A_m\sin\alpha x$

- Integration constants

$$C_1 = \frac{A_m + B_m}{\alpha^2}\frac{\sinh\alpha c + \alpha c\cosh\alpha c}{\sinh 2\alpha c + 2\alpha c}$$

$$C_2 = -\frac{A_m - B_m}{\alpha^2}\frac{\cosh\alpha c + \alpha c\sinh\alpha c}{\sinh 2\alpha c - 2\alpha c}$$

$$C_3 = \frac{A_m - B_m}{\alpha^2}\frac{\alpha\cosh\alpha c}{\sinh 2\alpha c - 2\alpha c}$$

$$C_4 = -\frac{A_m + B_m}{\alpha^2}\frac{\alpha\sinh\alpha c}{\sinh 2\alpha c + 2\alpha c}$$

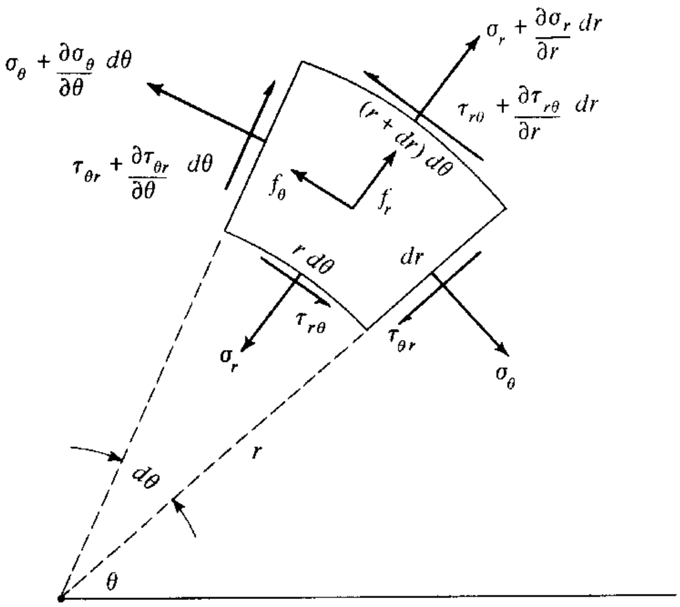
- Fourier coefficients

$$A_m = \int_{-l}^{l} q_u(x)\sin\frac{m\pi x}{l}dx \quad , \quad B_m = \int_{-l}^{l} q_b(x)\sin\frac{m\pi x}{l}dx$$

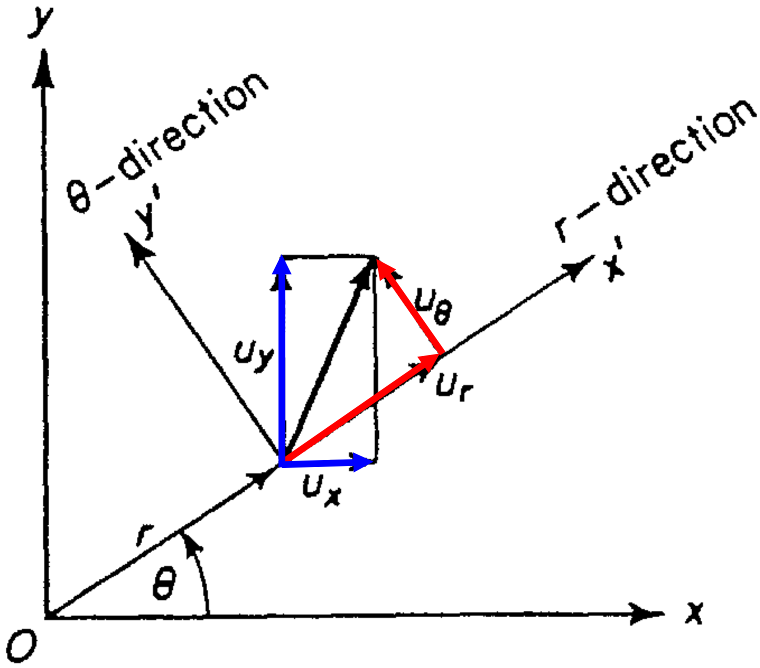
Refer to pp. 53-63 of *Theory of Elasticity* by Timoshenko.

Chapter 6

Plane Problems in Polar Coordinate System



6.1. Polar Coordinate system



- Basics

$$x = r\cos\theta \quad , \quad \theta = \tan^{-1}\frac{y}{x} \qquad z = z$$
$$y = r\sin\theta \quad , \quad r^2 = x^2 + y^2$$

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta \qquad , \qquad \frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta$$
$$\frac{\partial \theta}{\partial x} = -\frac{y}{r^2} = -\frac{\sin\theta}{r} \qquad \frac{\partial \theta}{\partial y} = \frac{x}{r^2} = \frac{\cos\theta}{r}$$

$$\frac{\partial()}{\partial x} = \frac{\partial()}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial()}{\partial \theta}\frac{\partial \theta}{\partial x} = \cos\theta\frac{\partial()}{\partial r} - \frac{\sin\theta}{r}\frac{\partial()}{\partial \theta}$$
$$\frac{\partial()}{\partial y} = \frac{\partial()}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial()}{\partial \theta}\frac{\partial \theta}{\partial y} = \sin\theta\frac{\partial()}{\partial r} + \frac{\cos\theta}{r}\frac{\partial()}{\partial \theta}$$

- Transformation matrix

$$\beta = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Stress

$$\sigma^c = \beta^T \sigma^p \beta = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{rr} & \sigma_{r\theta} & \sigma_{rz} \\ \sigma_{\theta r} & \sigma_{\theta\theta} & \sigma_{\theta z} \\ \sigma_{zr} & \sigma_{z\theta} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\sigma_{xx} = \sigma_{rr}\cos^2\theta + \sigma_{\theta\theta}\sin^2\theta - \sigma_{r\theta}\sin 2\theta$$

$$\sigma_{yy} = \sigma_{rr}\sin^2\theta + \sigma_{\theta\theta}\cos^2\theta + \sigma_{r\theta}\sin 2\theta$$

$$\sigma_z = \sigma_z$$

$$\sigma_{xy} = (\sigma_{rr} - \sigma_{\theta\theta})\sin\theta\cos\theta + \sigma_{r\theta}(\cos^2\theta - \sin^2\theta)$$

$$\sigma_{zx} = \sigma_{zr}\cos\theta - \sigma_{z\theta}\sin\theta$$

$$\sigma_{zy} = \sigma_{zr}\sin\theta + \sigma_{z\theta}\cos\theta$$

- Equilibrium Equation

$$\frac{\partial \sigma_{xx}}{\partial x} = \frac{\partial(\sigma_{rr}\cos^2\theta + \sigma_{\theta\theta}\sin^2\theta - \sigma_{r\theta}\sin 2\theta)}{\partial x}$$

$$= ??$$

$$\frac{\partial \sigma_{xy}}{\partial y} = \frac{\partial((\sigma_{rr} - \sigma_{\theta\theta})\sin\theta\cos\theta + \sigma_{r\theta}(\cos^2\theta - \sin^2\theta))}{\partial y}$$

$$= ??$$

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + b_x = ??$$

- Physical Derivation of the Equilibrium Equation.

$$\sum F_r = (\sigma_{rr} + \frac{\partial \sigma_{rr}}{\partial r}dr)(r+dr)d\theta - \sigma_{rr} r d\theta - (\sigma_{\theta\theta} + \frac{\partial \sigma_{\theta\theta}}{\partial \theta}d\theta + \sigma_{\theta\theta})dr\sin\frac{d\theta}{2} +$$

$$(\sigma_{r\theta} + \frac{\partial \sigma_{r\theta}}{\partial \theta}d\theta - \sigma_{r\theta})dr\cos\frac{d\theta}{2} + b_r r dr d\theta = 0$$

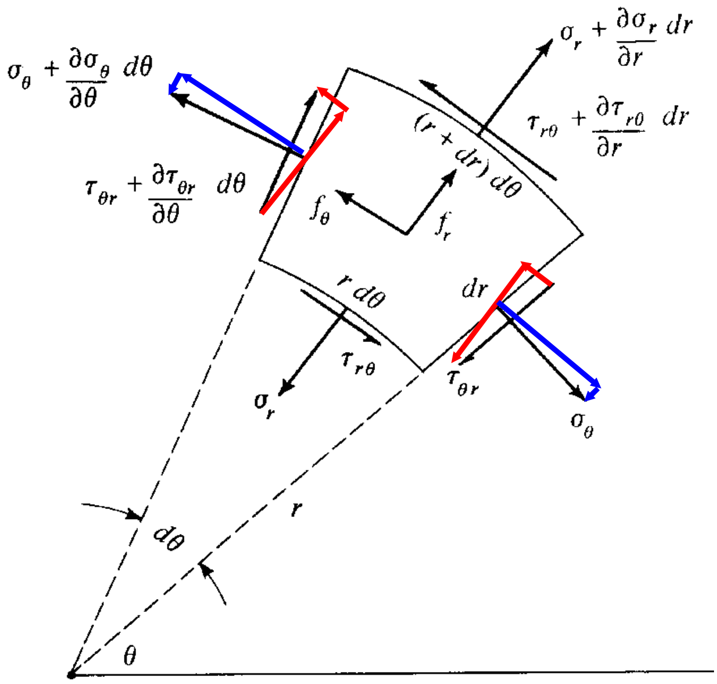
$$\sum F_\theta = ((\sigma_{\theta\theta} + \frac{\partial \sigma_{\theta\theta}}{\partial \theta}d\theta - \sigma_{\theta\theta})dr\cos\frac{d\theta}{2} + (\sigma_{r\theta} + \frac{\partial \sigma_{r\theta}}{\partial r}dr)(r+dr)d\theta - \sigma_{r\theta} r d\theta +$$

$$(\sigma_{r\theta} + \frac{\partial \sigma_{r\theta}}{\partial \theta}d\theta + \sigma_{r\theta})dr\sin\frac{d\theta}{2} + b_\theta r dr d\theta = 0$$

By neglecting higher order terms,

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) + b_r = 0$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + 2\frac{\sigma_{r\theta}}{r} + b_\theta = 0$$



- Stress Functions in Cartesian Coordinate System

$$\sigma_{xx} - \psi = \frac{\partial^2 \phi}{\partial y^2}\ ,\ \sigma_{yy} - \psi = \frac{\partial^2 \phi}{\partial x^2}\ ,\ \sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$$

- Second Derivatives

$$\frac{\partial}{\partial x}\frac{\partial ()}{\partial x} = \frac{\partial}{\partial x}(\cos\theta\frac{\partial ()}{\partial r} - \frac{\sin\theta}{r}\frac{\partial ()}{\partial \theta})$$
$$= ??$$
$$\frac{\partial}{\partial y}\frac{\partial ()}{\partial y} = \frac{\partial}{\partial y}(\sin\theta\frac{\partial ()}{\partial r} + \frac{\cos\theta}{r}\frac{\partial ()}{\partial \theta})$$
$$= ??$$
$$\frac{\partial}{\partial x}\frac{\partial ()}{\partial y} = \frac{\partial}{\partial x}(\sin\theta\frac{\partial ()}{\partial r} + \frac{\cos\theta}{r}\frac{\partial ()}{\partial \theta})$$
$$= ??$$

- Laplace Operator

$$\frac{\partial^2 ()}{\partial x^2} + \frac{\partial^2 ()}{\partial y^2} = \frac{\partial^2 ()}{\partial r^2} + \frac{1}{r}\frac{\partial ()}{\partial r} + \frac{1}{r^2}\frac{\partial^2 ()}{\partial \theta^2}$$

- Stress Function

$$\sigma_{rr} = \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + \sigma_{xy}\sin 2\theta$$
$$= \frac{\partial^2\phi}{\partial y^2}\cos^2\theta + \frac{\partial^2\phi}{\partial x^2}\sin^2\theta - \frac{\partial^2\phi}{\partial x\partial y}\sin 2\theta$$
$$= ??$$
$$\sigma_{\theta\theta} = \sigma_{xx}\sin^2\theta + \sigma_{yy}\cos^2\theta - \sigma_{xy}\sin 2\theta$$
$$= \frac{\partial^2\phi}{\partial y^2}\sin^2\theta + \frac{\partial^2\phi}{\partial x^2}\cos^2\theta + \frac{\partial^2\phi}{\partial x\partial y}\sin 2\theta$$
$$= ??$$
$$\sigma_{r\theta} = (\sigma_{yy} - \sigma_{xx})\sin\theta\cos\theta + \sigma_{xy}(\cos^2\theta - \sin^2\theta)$$
$$= (\frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial y^2})\sin\theta\cos\theta - \frac{\partial^2\phi}{\partial x\partial y}(\cos^2\theta - \sin^2\theta)$$
$$= ??$$

$$\sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$$
$$\sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2}$$
$$\sigma_{r\theta} = \frac{1}{r^2}\frac{\partial\phi}{\partial\theta} - \frac{1}{r}\frac{\partial^2\phi}{\partial r\partial\theta} = -\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial\phi}{\partial\theta})$$

- Displacement

$$u_r = u_x \cos\theta + u_y \sin\theta$$
$$u_\theta = -u_x \sin\theta + u_y \cos\theta$$
$$u_z = u_z$$

- Strain

$$\varepsilon^c = \beta^T \varepsilon^p \beta = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_{rr} & \varepsilon_{r\theta} & \varepsilon_{rz} \\ \varepsilon_{\theta r} & \varepsilon_{\theta\theta} & \varepsilon_{\theta z} \\ \varepsilon_{zr} & \varepsilon_{z\theta} & \varepsilon_{zz} \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\varepsilon_{xx} = \varepsilon_{rr}\cos^2\theta + \varepsilon_{\theta\theta}\sin^2\theta - \varepsilon_{r\theta}\sin 2\theta$$
$$\varepsilon_{yy} = \varepsilon_{rr}\sin^2\theta + \varepsilon_{\theta\theta}\cos^2\theta + \varepsilon_{r\theta}\sin 2\theta$$
$$\varepsilon_z = \varepsilon_z$$
$$\varepsilon_{xy} = (\varepsilon_{rr} - \varepsilon_{\theta\theta})\sin\theta\cos\theta + \varepsilon_{r\theta}(\cos^2\theta - \sin^2\theta)$$
$$\varepsilon_{zx} = \varepsilon_{zr}\cos\theta - \varepsilon_{z\theta}\sin\theta$$
$$\varepsilon_{zy} = \varepsilon_{zr}\sin\theta + \varepsilon_{z\theta}\cos\theta$$

- By chain rule

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} = (\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta})(u_r\cos\theta - u_\theta\sin\theta)$$
$$= \cos^2\theta\frac{\partial u_r}{\partial r} + \sin^2\theta(\frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta}) - \frac{1}{2}\sin 2\theta(\frac{\partial u_\theta}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r})$$
$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}\ ,\ \varepsilon_{\theta\theta} = (\frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta})\ ,\ \varepsilon_{r\theta} = \frac{1}{2}(\frac{\partial u_\theta}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r})$$

- Geometric derivation of strain components

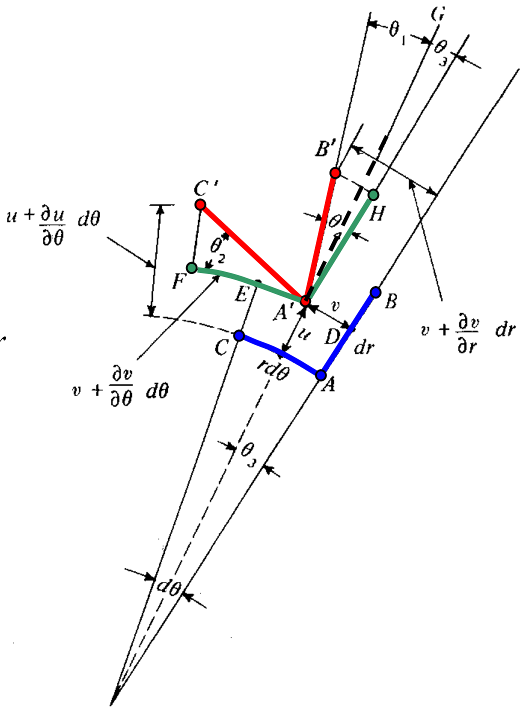
$$u = u_r,\ \ v = u_\theta$$

$$\varepsilon_{rr} = \frac{A'B' - AB}{AB} \approx \frac{A'H - AB}{AB}$$

$$AB = dr$$

$$A'H = r + dr + u_r + \frac{\partial u_r}{\partial r}dr - (r + u_r) = dr + \frac{\partial u_r}{\partial r}dr$$

$$\varepsilon_{rr} = \frac{dr + (\partial u_r/\partial r)dr - dr}{dr} = \frac{\partial u_r}{\partial r}$$



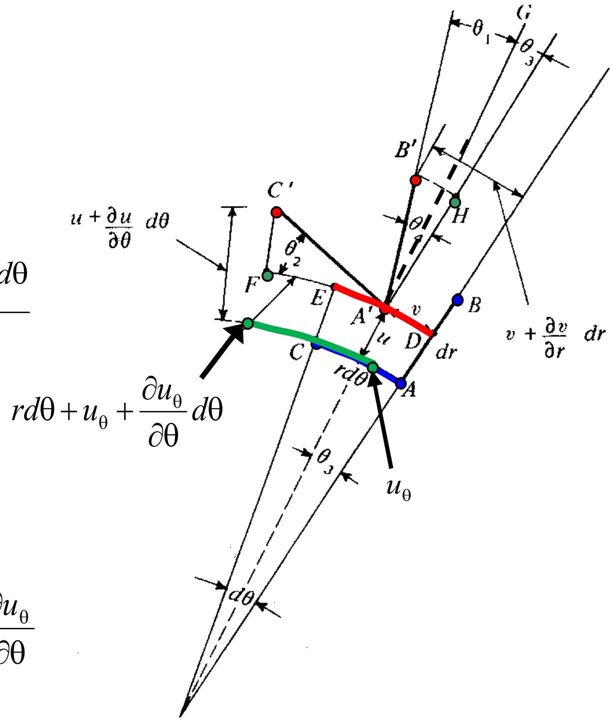
$u_\theta = 0$, $u_r \neq 0$ (Red line)

$$\varepsilon_{\theta\theta} = \frac{DE - AC}{AC} = \frac{(r + u_r)d\theta - rd\theta}{rd\theta} = \frac{u_r}{r}$$

$u_\theta \neq 0$, $u_r = 0$ (Green line)

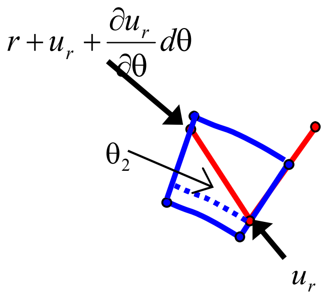
$$\varepsilon_{\theta\theta} = \frac{A''F'' - AC}{AC} = \frac{(rd\theta + u_\theta + \frac{\partial u_\theta}{\partial\theta}d\theta - u_\theta) - rd\theta}{rd\theta}$$

$$= \frac{1}{r}\frac{\partial u_\theta}{\partial\theta}$$



$u_\theta \neq 0$, $u_r \neq 0$

$$\varepsilon_{\theta\theta} = \frac{(r + u_r)d\theta + (\partial u_\theta / \partial\theta)d\theta - rd\theta}{rd\theta} = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta}$$



$u_\theta = 0$, $u_r \neq 0$

$$\theta_2 = \frac{FC'}{A'F} = \frac{(r + u_r + \partial u_r / \partial\theta d\theta) - (r + u_r)}{rd\theta} = \frac{1}{r}\frac{\partial u_r}{\partial\theta}$$

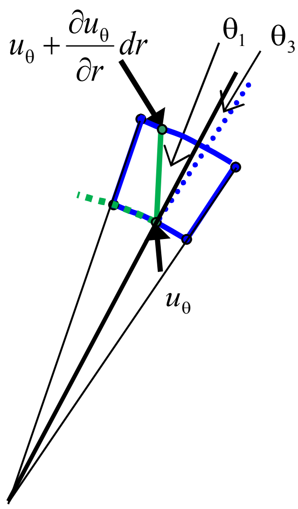
$u_\theta \neq 0$, $u_r = 0$

$$\theta_4 = \theta_1 + \theta_3 = \frac{u_\theta + \partial u_r / \partial r dr - u_\theta}{dr} = \frac{\partial u_r}{\partial r}, \quad \theta_3 = \frac{u_\theta}{r}$$

$u_\theta \neq 0$, $u_r \neq 0$

$$\gamma_{r\theta} = \theta_2 + \theta_1 = (\theta_2 + \theta_4 - \theta_3) = \frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}$$

$$\varepsilon_{r\theta} = \frac{1}{2}(\theta_2 + \theta_4 - \theta_3) = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)$$



6.2. Governing Equation and Boundary Conditions

- Equilibrium

$$\frac{\partial\sigma_{rr}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta}+\frac{1}{r}(\sigma_{rr}-\sigma_{\theta\theta})+b_r=0$$

$$\frac{\partial\sigma_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta}+2\frac{\sigma_{r\theta}}{r}+b_\theta=0$$

- Stress Function

$$\sigma_{rr}=\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$$

$$\sigma_{\theta\theta}=\frac{\partial^2\phi}{\partial r^2}$$

$$\sigma_{r\theta}=\frac{1}{r^2}\frac{\partial\phi}{\partial\theta}-\frac{1}{r}\frac{\partial^2\phi}{\partial r\partial\theta}=-\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial\phi}{\partial\theta})$$

- Compatibility

$$\frac{\partial^2()}{\partial x^2}+\frac{\partial^2()}{\partial y^2}=\frac{\partial^2()}{\partial r^2}+\frac{1}{r}\frac{\partial()}{\partial r}+\frac{1}{r^2}\frac{\partial^2()}{\partial\theta^2}$$

$$\nabla^4\phi=(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2})(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2})\phi=(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2})(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2})\phi=0$$

- Strain Components

$$\varepsilon_{rr}=\frac{\partial u_r}{\partial r}\quad,\quad\varepsilon_{\theta\theta}=\frac{u_r}{r}+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\quad,\quad\varepsilon_{r\theta}=\frac{1}{2}(\frac{1}{r}\frac{\partial u_r}{\partial\theta}+\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r})$$

- Constitutive Law

$$\sigma^p_{ij}=C_{ijkl}\varepsilon^p_{kl}$$

- Displacement Boundary Conditions

$$u^c_i=\bar{u}^c_i\rightarrow u^p_i=\beta_{ij}u^c_j=\beta_{ij}\bar{u}^c_j=\bar{u}^p_i\rightarrow\mathbf{u}^p=\bar{\mathbf{u}}^p$$

- Cauchy's Relation

$$T^c_i=\sigma^c_{ij}n^c_j\rightarrow T^p_i=\beta_{ij}T^c_j=\beta_{ij}\sigma^c_{jk}n^c_k=\beta_{ij}\sigma^c_{jn}\beta_{mk}\beta_{mn}n^c_k=\beta_{ij}\sigma^c_{jn}\beta_{mn}\beta_{mk}n^c_k=\sigma^p_{jm}n^p_m$$

$$\mathbf{T}^p=\boldsymbol{\sigma}^p\mathbf{n}^p$$

- Traction Boundary Condition

$$T^c_i=\bar{T}^c_i\rightarrow T^p_i=\beta_{ij}T^c_j=\beta_{ij}\bar{T}^c_j=\bar{T}^p_i\rightarrow\mathbf{T}^p=\bar{\mathbf{T}}^p$$

6.3. Solutions in the Polar Coordinate System

Compatibility Equation

- Compatibility Equation

$$(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2})(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2})\phi=0$$

- In case ϕ is a function of r only

$$(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r})(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r})\phi=0\rightarrow\frac{d^4\phi}{dr^4}+\frac{2}{r}\frac{d^3\phi}{dr^3}-\frac{1}{r^2}\frac{d^2\phi}{dr^2}+\frac{1}{r^3}\frac{d\phi}{dr}=0$$

- Solution of the Compatibility Equation

$$r=e^t\geq 0\rightarrow dr=e^t dt$$

$$\frac{d\phi}{dr}=\frac{d\phi}{dt}\frac{dt}{dr}=\frac{1}{r}\frac{d\phi}{dt}$$
$$\frac{d^2\phi}{dr^2}=\frac{d}{dt}(\frac{1}{r}\frac{d\phi}{dt})\frac{dt}{dr}=\frac{1}{r^2}(\frac{d^2\phi}{dt^2}-\frac{d\phi}{dt})$$
$$\frac{d^3\phi}{dr^3}=\frac{d}{dt}(\frac{1}{r^2}(\frac{d^2\phi}{dt^2}-\frac{d\phi}{dt}))\frac{dt}{dr}=\frac{1}{r^3}(\frac{d^3\phi}{dt^3}-3\frac{d^2\phi}{dt^2}+2\frac{d\phi}{dt})$$
$$\frac{d^4\phi}{dr^4}=\frac{1}{r^4}(\frac{d^4\phi}{dt^4}-6\frac{d^3\phi}{dt^3}+11\frac{d^2\phi}{dt^2}-6\frac{d\phi}{dt})$$

$$\frac{d^4\phi}{dr^4}+\frac{2}{r}\frac{d^3\phi}{dr^3}-\frac{1}{r^2}\frac{d^2\phi}{dr^2}+\frac{1}{r^3}\frac{d\phi}{dr}=$$
$$\frac{1}{r^4}(\frac{d^4\phi}{dt^4}-6\frac{d^3\phi}{dt^3}+11\frac{d^2\phi}{dt^2}-6\frac{d\phi}{dt})+\frac{2}{r}\frac{1}{r^3}(\frac{d^3\phi}{dt^3}-3\frac{d^2\phi}{dt^2}+2\frac{d\phi}{dt})-\frac{1}{r^2}\frac{1}{r^2}(\frac{d^2\phi}{dt^2}-\frac{d\phi}{dt})+\frac{1}{r^3}\frac{1}{r}\frac{d\phi}{dt}=$$
$$\frac{1}{r^4}(\frac{d^4\phi}{dt^4}-4\frac{d^3\phi}{dt^3}+4\frac{d^2\phi}{dt^2})=0$$

$$\phi=At+Bte^{2t}+Ce^{2t}+D=A\ln r+Br^2\ln r+Cr^2+D$$

- Stress Components

$$\sigma_{rr}=\frac{1}{r}\frac{d\phi}{dr}=\frac{A}{r^2}+B(1+2\ln r)+2C$$
$$\sigma_{\theta\theta}=\frac{d^2\phi}{dr^2}=-\frac{A}{r^2}+B(3+2\ln r)+2C$$
$$\sigma_{r\theta}=-\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial\phi}{\partial\theta})=0$$

- If no hole exists at the origin of coordinates, A=B=0
- The term associated with B is not a single-valued function.

- Displacement (plane stress)

$$\varepsilon_{rr} = \frac{1}{E}(\sigma_{rr} - \nu\sigma_{\theta\theta}) = \frac{\partial u_r}{\partial r} = \frac{1}{E}[\frac{(1+\nu)A}{r^2} + 2(1-\nu)B\ln r + (1-3\nu)B + 2(1-\nu)C]$$

$$u_r = \frac{1}{E}[-\frac{(1+\nu)A}{r} + 2(1-\nu)Br\ln r - (1+\nu)Br + 2(1-\nu)Cr] + f(\theta)$$

$$\varepsilon_{\theta\theta} = \frac{1}{E}(\sigma_{\theta\theta} - \nu\sigma_{rr}) = \frac{u_r}{r} + \frac{\partial u_\theta}{r\partial\theta} = \frac{1}{E}[-\frac{(1+\nu)A}{r^2} + 2(1-\nu)B\ln r + (3-\nu)B + 2(1-\nu)C]$$

$$\frac{\partial u_\theta}{\partial\theta} = \frac{4Br}{E} - f(\theta) \rightarrow u_\theta = \frac{4Br\theta}{E} - \int f(\theta)d\theta + f_1(r)$$

$$\varepsilon_{r\theta} = \frac{1}{2}(\frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}) = 0 \rightarrow \frac{1}{r}\frac{\partial f(\theta)}{\partial\theta} + \frac{\partial f_1(r)}{\partial r} + \frac{1}{r}\int f(\theta)d\theta - \frac{1}{r}f_1(r) = 0$$

$$\frac{\partial f(\theta)}{\partial\theta} + \int f(\theta)d\theta = p \rightarrow \frac{\partial^2 f(\theta)}{\partial\theta^2} + f(\theta) = 0 \rightarrow f(\theta) = H\sin\theta + K\cos\theta$$

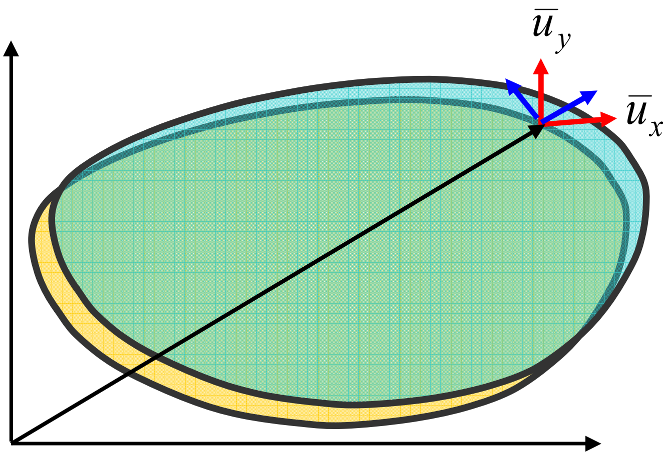
$$r\frac{\partial f_1(r)}{\partial r} - f_1(r) = -p \rightarrow f_1(r) = Fr - p$$

Substitution of the above solutions into the original Eq., $p = 0$.

$$u_r = \frac{1}{E}[-\frac{(1+\nu)A}{r} + 2(1-\nu)Br\ln r - (1+\nu)Br + 2(1-\nu)Cr] + H\sin\theta + K\cos\theta$$

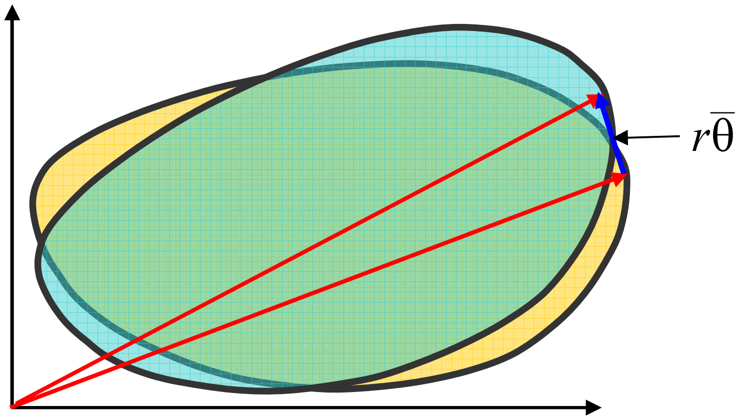
$$u_\theta = \frac{4Br\theta}{E} + H\cos\theta - K\sin\theta + Fr$$

– Rigid body translation



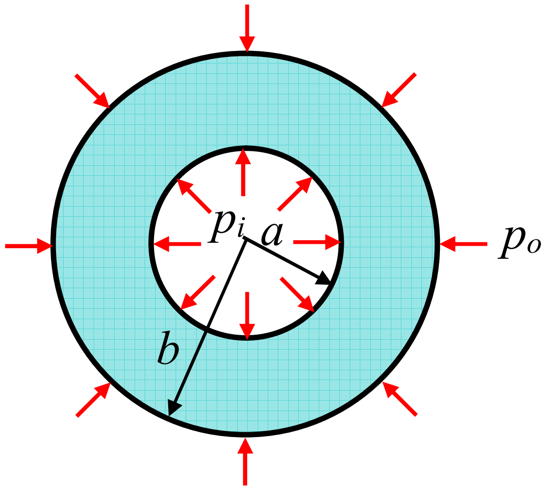
$$\bar{u}_r = \bar{u}_x\cos\theta + \bar{u}_y\sin\theta,\ \bar{u}_\theta = -\bar{u}_x\sin\theta + \bar{u}_y\cos\theta \rightarrow K = \bar{u}_x\ ,\ \ H = \bar{u}_y$$

– Rigid body rotation



$$\overline{u}_r = 0 \ , \ \ \overline{u}_\theta = r\overline{\theta} \rightarrow F = \overline{\theta}$$

6.4. Thick Pipe Problem



- Traction BC
  - On $r = a$: $T_i = \sigma_{ij} n_j \rightarrow \sigma_{rr} = -p_i$
  - On $r = b$: $T_i = \sigma_{ij} n_j \rightarrow \sigma_{rr} = -p_o$

$$\begin{aligned} \sigma_{rr}(a) &= \frac{A}{a^2} + 2C = -p_i \\ \sigma_{rr}(b) &= \frac{A}{b^2} + 2C = -p_o \end{aligned} \quad \rightarrow \quad \begin{aligned} A &= \frac{a^2 b^2 (p_o - p_i)}{b^2 - a^2} \\ 2C &= \frac{p_i a^2 - p_o b^2}{b^2 - a^2} \end{aligned}$$

- Stress components

$$\sigma_{rr} = \frac{a^2 b^2 (p_o - p_i)}{b^2 - a^2} \frac{1}{r^2} + \frac{p_i a^2 - p_o b^2}{b^2 - a^2}$$

$$\sigma_{\theta\theta} = -\frac{a^2 b^2 (p_o - p_i)}{b^2 - a^2} \frac{1}{r^2} + \frac{p_i a^2 - p_o b^2}{b^2 - a^2}$$

- Displacement

$$u_r = \frac{1}{E}\left[-\frac{a^2 b^2 (p_o - p_i)}{b^2 - a^2} \frac{(1+\nu)}{r} + \frac{p_i a^2 - p_o b^2}{b^2 - a^2}(1-\nu) r\right], \quad u_\theta = 0$$

- End Conditions

  – Plane strain

$$\varepsilon_{zz} = 0$$

$$\sigma_{zz} = \nu(\sigma_{rr} + \sigma_{\theta\theta}) = 2\nu \frac{p_i a^2 - p_o b^2}{b^2 - a^2}$$

  – Plane stress (open end)

$$\sigma_{zz} = 0$$

$$\varepsilon_{zz} = -\frac{\nu(\sigma_{rr} + \sigma_{\theta\theta})}{E} = -2\frac{\nu}{E}\frac{p_i a^2 - p_o b^2}{b^2 - a^2}$$

  – Closed End with a Rigid End Plate

$$F_z = \pi(b^2 - a^2)\sigma_{zz} = \pi(b^2 - a^2)(E\varepsilon_{zz} + \nu(\sigma_{rr} + \sigma_{\theta\theta}))$$

$$= \pi(b^2 - a^2)(E\varepsilon_{zz} + 2\nu \frac{p_i a^2 - p_o b^2}{b^2 - a^2}) = \pi(a^2 p_i - b^2 p_o)$$

$$\varepsilon_{zz} = \frac{(1-2\nu)a^2 p_i - (1-2\nu)b^2 p_o}{E(b^2 - a^2)}$$

$$\sigma_{zz} = \frac{a^2 p_i - b^2 p_o}{b^2 - a^2}$$

- $p_o = 0$

  – Stress : $\sigma_{rr} = \frac{a^2 p_i}{b^2 - a^2}(1 - \frac{b^2}{r^2})$, $\sigma_{\theta\theta} = \frac{a^2 p_i}{b^2 - a^2}(1 + \frac{b^2}{r^2})$

$$(\sigma_{\theta\theta})_{\max} = \frac{p_i(b^2 + a^2)}{b^2 - a^2} > p_i$$

  – Displacement : $u_r = \frac{p_i b}{E}\frac{a^2}{b^2 - a^2}[(1+\nu)\frac{b}{r} + (1-\nu)\frac{r}{b}]$

6.5. Special Cases

- Thin pipe with an internal pressure: $b-a=t<<b$

$$a=r_0-\frac{t}{2}\ ,\ \ b=r_0+\frac{t}{2}$$

$$\sigma_{rr}=\frac{a^2p_i}{b^2-a^2}(1-\frac{b^2}{r^2})\approx 0$$

$$\sigma_{\theta\theta}=\frac{a^2p_i}{b^2-a^2}(1+\frac{b^2}{r^2})=\frac{r_0p_i}{t}$$

$$u_r=\frac{p_ib}{E}\frac{a^2}{b^2-a^2}[(1+\nu)\frac{b}{r}+(1-\nu)\frac{r}{b}]=\frac{p_ir_0^2}{Et}$$

- Infinite Region pipe with an internal pressure: $b\rightarrow\infty$

$$\sigma_{rr}=\frac{a^2p_i}{b^2-a^2}(1-\frac{b^2}{r^2})=\frac{a^2p_i}{1-(a/b)^2}(\frac{1}{b^2}-\frac{1}{r^2})$$

$$\sigma_{\theta\theta}=\frac{a^2p_i}{b^2-a^2}(1+\frac{b^2}{r^2})=\frac{a^2p_i}{1-(a/b)^2}(\frac{1}{b^2}+\frac{1}{r^2})$$

$$u_r=\frac{p_ib}{E}\frac{a^2}{b^2-a^2}[(1+\nu)\frac{b}{r}+(1-\nu)\frac{r}{b}]=\frac{p_i}{E}\frac{a^2}{1-(a/b)^2}[(1+\nu)\frac{1}{r}+(1-\nu)\frac{r}{b^2}]$$

as $b\rightarrow\infty$,

$$\sigma_{rr}=-\frac{a^2p_i}{r^2}\ ,\quad \sigma_{\theta\theta}=\frac{a^2p_i}{r^2}\ ,\quad u_r=(1+\nu)a\frac{p_i}{E}\frac{a}{r}$$

- Infinite Region pipe with a Lining

  – Region 1

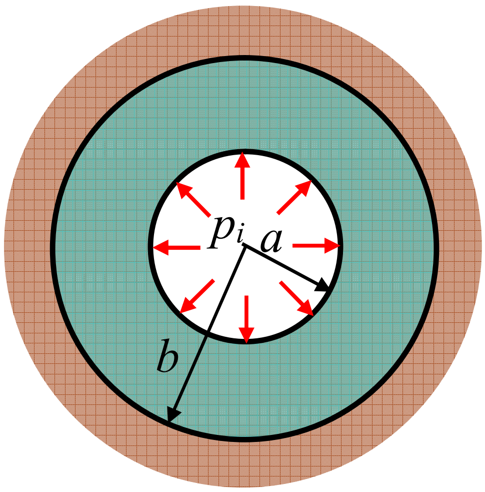
$$\sigma_{rr}=\frac{a^2b^2(p_o-p_i)}{b^2-a^2}\frac{1}{r^2}+\frac{p_ia^2-p_ob^2}{b^2-a^2}$$

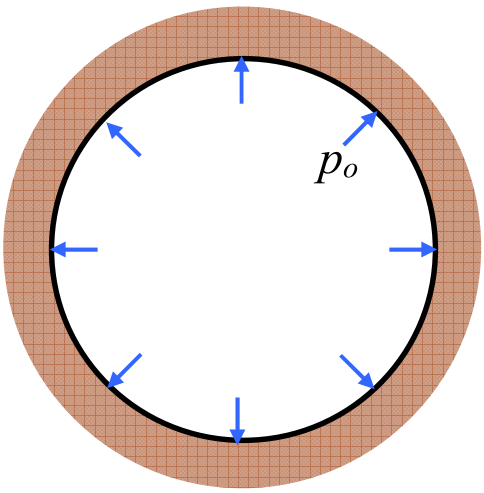
$$\sigma_{\theta\theta}=-\frac{a^2b^2(p_o-p_i)}{b^2-a^2}\frac{1}{r^2}+\frac{p_ia^2-p_ob^2}{b^2-a^2}$$

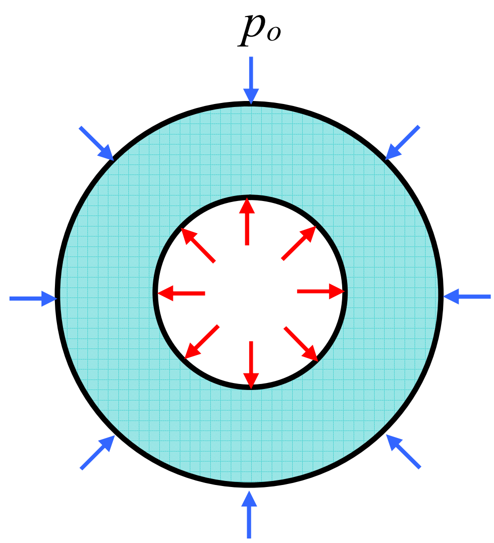
$$u_r^1=\frac{1}{E_1}[-\frac{a^2b^2(p_o-p_i)}{b^2-a^2}\frac{(1+\nu_1)}{r}+\frac{p_ia^2-p_ob^2}{b^2-a^2}(1-\nu_1)r]\ ,\ u_\theta=0$$

  – Region 2

$$\sigma_{rr}=-\frac{b^2p_o}{r^2}\ ,\quad \sigma_{\theta\theta}=\frac{b^2p_0}{r^2}\ ,\quad u_r^2=(1+\nu_2)b\frac{p_o}{E_2}\frac{b}{r}$$







– Compatibility

$$u_r^2(b) = u_r^1(b)$$

$$(1+\nu_2)b\frac{p_o}{E_2} = \frac{1}{E_1}[-\frac{a^2b^2(p_o - p_i)}{b^2-a^2}\frac{(1+\nu_1)}{b} + \frac{p_i a^2 - p_o b^2}{b^2-a^2}(1-\nu_1)b]$$

$$(1+\nu_2)b\frac{p_o}{E_2} + \frac{1}{E_1}[\frac{a^2bp_o}{b^2-a^2}(1+\nu_1) + \frac{p_o b^3}{b^2-a^2}(1-\nu_1)] = \frac{1}{E_1}[\frac{a^2bp_i}{b^2-a^2}(1+\nu_1) + \frac{p_i a^2 b}{b^2-a^2}(1-\nu_1)]$$

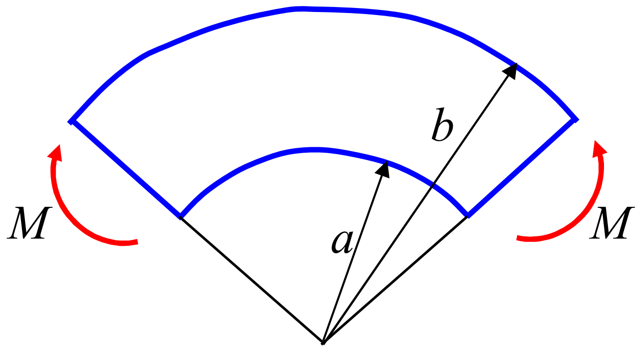
$$(1+\nu_2)b\frac{p_o}{E_2} + \frac{1}{E_1}[\frac{a^2bp_o}{b^2-a^2}(1+\nu_1) + \frac{p_o b^3}{b^2-a^2}(1-\nu_1)] = \frac{1}{E_1}\frac{2a^2bp_i}{b^2-a^2}$$

$$p_o = E_2\frac{2a^2p_i}{E_1(1+\nu_2)(b^2-a^2) + E_2(a^2+b^2+\nu_1(a^2-b^2))}$$

– A thin lining

$$p_o = E_2\frac{(r_0 - t)p_i}{E_1(1+\nu_2)t + E_2(r_0 - \nu_1 t)} \approx \frac{r_0 - t}{r_0 + t}p_i$$

6.6. Curved Beam



- Boundary Condition
  - On $r = a$ and b : $T_i = \sigma_{ij} n_j = 0 \rightarrow \sigma_{rr} = \sigma_{r\theta} = 0$
  - On $\theta = 0$: $T_r = \sigma_{r\theta} n_\theta = 0$ and $\overline{T}_\theta = \sigma_{\theta\theta} n_\theta = \sigma_{\theta\theta}$

- Applied Moment

$$M = \int_a^b \overline{T}_\theta r dr = -\int_a^b \sigma_{\theta\theta} r dr \quad , \quad \int_a^b \overline{T}_\theta dr = \int_a^b \sigma_{\theta\theta} dr = 0$$

- Stress Component

$$\sigma_{rr} = \frac{1}{r}\frac{d\phi}{dr} = \frac{A}{r^2} + B(1 + 2\ln r) + 2C$$

$$\sigma_{\theta\theta} = \frac{d^2\phi}{dr^2} = -\frac{A}{r^2} + B(3 + 2\ln r) + 2C$$

- Application BC on $r = a$ and b

$$\frac{A}{a^2} + B(1 + 2\ln a) + 2C = 0$$

$$\frac{A}{b^2} + B(1 + 2\ln b) + 2C = 0$$

- Moment Condition

$$M = -\int_a^b \sigma_{\theta\theta} r dr = -\int_a^b \frac{d^2\phi}{dr^2} r dr = -\frac{d\phi}{dr} r \Big|_a^b + \int_a^b \frac{d\phi}{dr} dr$$

$$= -\frac{d\phi}{dr} r \Big|_a^b + \phi\Big|_a^b = -r^2 \sigma_{rr}\Big|_a^b + \phi\Big|_a^b = \phi(b) - \phi(a)$$

$$\phi = A\ln r + Br^2 \ln r + Cr^2 + D$$

$$M = A\ln\frac{b}{a} + B(b^2 \ln b - a^2 \ln a) + C(b^2 - a^2)$$

- Solution

$$A = -\frac{4M}{N} a^2 b^2 \ln\frac{b}{a}$$
$$B = -\frac{2M}{N}(b^2 - a^2)$$
$$C = \frac{M}{N}(b^2 - a^2 + 2(b^2 \ln b - a^2 \ln a))$$

where $N = (b^2 - a^2)^2 - 4a^2 b^2 (\ln\frac{b}{a})^2$.

- Stress

$$\sigma_{rr} = -\frac{4M}{N}(\frac{a^2 b^2}{r^2}\ln\frac{b}{a} + b^2 \ln\frac{r}{b} + a^2 \ln\frac{a}{r})$$
$$\sigma_{\theta\theta} = -\frac{4M}{N}(-\frac{a^2 b^2}{r^2}\ln\frac{b}{a} + b^2 \ln\frac{r}{b} + a^2 \ln\frac{a}{r} + b^2 - a^2)$$

- Displacement

$$u_r = \frac{1}{E}[-\frac{(1+\nu)A}{r} + 2(1-\nu)Br\ln r - (1+\nu)Br + 2(1-\nu)Cr] + H\sin\theta + K\cos\theta$$
$$u_\theta = \frac{4Br\theta}{E} + H\cos\theta - K\sin\theta + Fr$$

- Displacement B.C.

$$u_r = u_\theta = \frac{\partial u_\theta}{\partial r} = 0 \quad \text{at} \quad \theta = 0 \quad \text{and} \quad r = (a+b)/2 \rightarrow F = H = 0$$

- A ring with cut out.

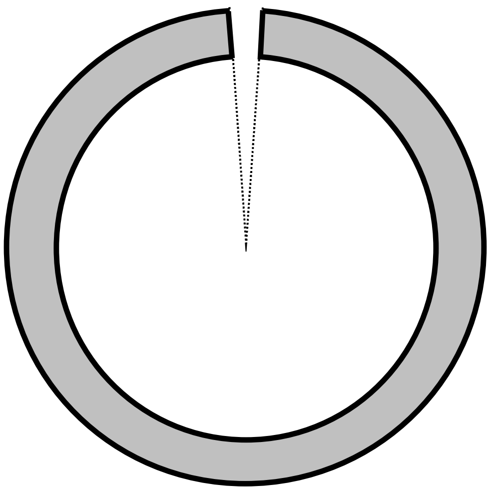
  – Cut out angle : α

  – Compatibility condition

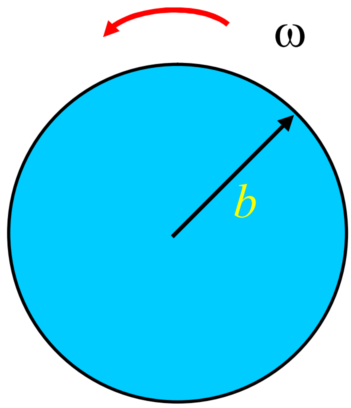
$$u_\theta(2\pi) = 8\pi\frac{Br}{E} = \alpha r \rightarrow B = \frac{\alpha E}{8\pi}$$

  – Moment required to close the ring

$$B = -\frac{2M}{N}(b^2 - a^2) = \frac{\alpha E}{8\pi}$$
$$\rightarrow M = -\frac{\alpha E N}{16\pi(b^2 - a^2)}$$



6.7. Rotating Disk



- Centrifugal force : $b_r = \rho\omega^2 r$
- Equilibrium Equation

$$\left.\begin{aligned} &\frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta}) + b_r = 0 \\ &\frac{\partial\sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta} + 2\frac{\sigma_{r\theta}}{r} + b_\theta = 0 \end{aligned}\right\} \rightarrow \frac{\partial r\sigma_{rr}}{\partial r} - \sigma_{\theta\theta} = -\rho\omega^2 r^2$$

- Stress and strain

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r} \quad , \quad \varepsilon_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} = \frac{u_r}{r}$$

$$\sigma_{rr} = \frac{E}{1-\nu^2}(\varepsilon_{rr} + \nu\varepsilon_{\theta\theta}) = \frac{E}{1-\nu^2}(\frac{\partial u_r}{\partial r} + \nu\frac{u_r}{r})$$

$$\sigma_{\theta\theta} = \frac{E}{1-\nu^2}(\nu\varepsilon_{rr} + \varepsilon_{\theta\theta}) = \frac{E}{1-\nu^2}(\nu\frac{\partial u_r}{\partial r} + \frac{u_r}{r})$$

- Equilibrium Equation in terms of displacement

$$r^2\frac{d^2u_r}{dr^2} + r\frac{du_r}{dr} - u_r = -\frac{1-v^2}{E}\rho\omega^2 r^3 \quad \text{(Euler-Cauchy Equation)}$$

- General Solution

$$u_r = \frac{1}{E}[(1-v)Cr - (1+v)C_1\frac{1}{r} - \frac{1-\nu^2}{8}\rho\omega^2 r^3]$$

$$\sigma_{rr} = \frac{E}{1-\nu^2}\left(\frac{\partial u_r}{\partial r} + \nu \frac{u_r}{r}\right)$$
$$= \frac{1}{1-\nu^2}[(1-\nu)C + (1+\nu)C_1 \frac{1}{r^2} - 3\frac{1-\nu^2}{8}\rho\omega^2 r^2 + \nu((1-\nu)C - (1+\nu)C_1 \frac{1}{r^2} - \frac{1-\nu^2}{8}\rho\omega^2 r^2)]$$
$$= \frac{1}{1-\nu^2}[(1-\nu^2)C + (1-\nu^2)C_1 \frac{1}{r^2} - (3+\nu)\frac{1-\nu^2}{8}\rho\omega^2 r^2]$$
$$= C + C_1 \frac{1}{r^2} - \frac{(3+\nu)}{8}\rho\omega^2 r^2$$
$$\sigma_{\theta\theta} = C - C_1 \frac{1}{r^2} - \frac{(1+3\nu)}{8}\rho\omega^2 r^2$$

- Solid disk

$$C_1 = 0 \ , \ (\sigma_{rr})_{r=b} = C - \frac{(3+\nu)}{8}\rho\omega^2 b^2 = 0 \rightarrow C = \frac{(3+\nu)}{8}\rho\omega^2 b^2$$
$$\sigma_{rr} = \frac{(3+\nu)}{8}\rho\omega^2(b^2 - r^2) \ , \ \sigma_{\theta\theta} = \frac{(3+\nu)}{8}\rho\omega^2 b^2 - \frac{(1+3\nu)}{8}\rho\omega^2 r^2$$
$$(\sigma_{rr})_{\max} = (\sigma_{\theta\theta})_{\max} = \frac{(3+\nu)}{8}\rho\omega^2 b^2 \quad \text{at } r = 0$$

- Disk with a circular hole at the center

$$\left.\begin{aligned}(\sigma_{rr})_{r=a} &= C + \frac{C_1}{a^2} - \frac{(3+\nu)}{8}\rho\omega^2 a^2 = 0\\ (\sigma_{rr})_{r=b} &= C + \frac{C_1}{b^2} - \frac{(3+\nu)}{8}\rho\omega^2 b^2 = 0\end{aligned}\right\} \rightarrow \left\{\begin{aligned}C &= \frac{(3+\nu)}{8}\rho\omega^2(a^2+b^2)\\ C_1 &= -\frac{(3+\nu)}{8}\rho\omega^2 a^2 b^2\end{aligned}\right.$$

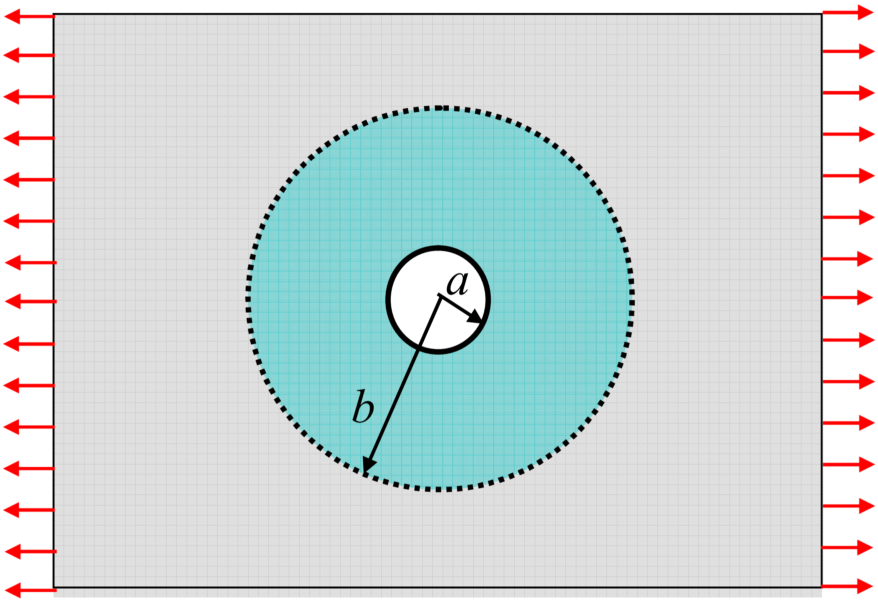
$$\sigma_{rr} = \frac{3+\nu}{8}\rho\omega^2\left(a^2 + b^2 - \frac{a^2b^2}{r^2} - r^2\right)$$
$$\sigma_{\theta\theta} = \frac{3+\nu}{8}\rho\omega^2\left(a^2 + b^2 + \frac{a^2b^2}{r^2} - \frac{1+3\nu}{3+\nu}r^2\right)$$
$$(\sigma_{rr})_{\max} = \frac{3+\nu}{8}\rho\omega^2(b-a)^2 \quad \text{at } r = \sqrt{ab}$$
$$(\sigma_{\theta\theta})_{\max} = \frac{3+\nu}{4}\rho\omega^2\left(b^2 + \frac{1-\nu}{3+\nu}a^2\right) \quad \text{at } r = a$$

- 7200 rpm 3.5 inch (8.9 cm) hard disk

$$(\sigma_{\theta\theta})_{\max} = \frac{3.3}{4}7850 \times 120^2(4.5^2 + \frac{1-0.3}{3+0.3}0.5^2) \times 10^{-4}$$
$$= 189342.0 kg/m\sec^2 = 189342.0 N/m^2 \approx 1.9 kg(f)/cm^2$$

- Solution by force method and Disk with two different material (homework)

6.8. Plate with Circular hole



- Stress on remote field

$$\sigma_{xx} = S \ , \quad \sigma_{yy} = 0 \ , \quad \sigma_{xy} = 0 \quad \text{(Cartesian)}$$

$$\sigma_{rr} = \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + \sigma_{xy}\sin 2\theta = \frac{S}{2}(1+\cos 2\theta)$$

$$\sigma_{\theta\theta} = \sigma_{xx}\sin^2\theta + \sigma_{yy}\cos^2\theta - \sigma_{xy}\sin 2\theta = \frac{S}{2}(1-\cos 2\theta)$$

$$\sigma_{r\theta} = (\sigma_{yy} - \sigma_{xx})\sin\theta\cos\theta + \sigma_{xy}(\cos^2\theta - \sin^2\theta) = -\frac{S}{2}\sin 2\theta$$

- Solution for the uniform stress (Thick pipe problem)

$$\left.\begin{aligned} \sigma_{rr} &= \frac{S}{2}\left(\frac{b^2}{b^2-a^2} - \frac{b^2}{b^2-a^2}\frac{a^2}{r^2}\right) \\ \sigma_{\theta\theta} &= \frac{S}{2}\left(\frac{b^2}{b^2-a^2} + \frac{b^2}{b^2-a^2}\frac{a^2}{r^2}\right) \end{aligned}\right\} \xrightarrow[b>>a]{} \left\{\begin{aligned} \sigma_{rr} &= \frac{S}{2}\left(1-\frac{a^2}{r^2}\right) \\ \sigma_{\theta\theta} &= \frac{S}{2}\left(1+\frac{a^2}{r^2}\right) \end{aligned}\right.$$

$$\sigma_{r\theta} = 0$$

- Solution for non-uniform stress field

  – BC on $r = b$: $\sigma_{rr} = \frac{S}{2}\cos 2\theta$, $\sigma_{r\theta} = -\frac{S}{2}\sin 2\theta$

  – Stress Function : $\phi = f(r)\cos 2\theta$

$$\sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} \ , \quad \sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2}$$

$$\sigma_{r\theta} = \frac{1}{r^2}\frac{\partial\phi}{\partial\theta} - \frac{1}{r}\frac{\partial^2\phi}{\partial r\partial\theta} = -\frac{\partial\phi}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)$$

- Compatibility Equation

$$\nabla^4\phi = (\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}+\frac{1}{r^2}\frac{d^2}{d\theta^2})(\frac{d^2 f}{dr^2}+\frac{1}{r}\frac{df}{dr}-\frac{4}{r^2}f)\cos 2\theta = 0$$

$$F = \frac{d^2 f}{dr^2}+\frac{1}{r}\frac{df}{dr}-\frac{4}{r^2}f$$

$$\nabla^4\phi = (\frac{d^2 F}{dr^2}+\frac{1}{r}\frac{dF}{dr}-\frac{4}{r^2}F)\cos 2\theta = 0$$

$$\frac{d^2 F}{dr^2}+\frac{1}{r}\frac{dF}{dr}-\frac{4}{r^2}F = 0$$

$$F = B'r^2 + D'\frac{1}{r^2} \rightarrow \frac{d^2 f}{dr^2}+\frac{1}{r}\frac{df}{dr}-\frac{4}{r^2}f = B'r^2 + D'\frac{1}{r^2}$$

$$f = Ar^2 + C\frac{1}{r^2} + f_p$$

$$f_p = Kr^4 + H \rightarrow 12Kr^2 + 4Kr^2 - \frac{4}{r^2}(Kr^4 + H) = B'r^2 + D'\frac{1}{r^2}$$

$$K = \frac{B'}{12} \;,\quad H = -\frac{D'}{4}$$

$$f = Ar^2 + \frac{B'}{12}r^4 + C\frac{1}{r^2} - \frac{D'}{4} = Ar^2 + Br^4 + C\frac{1}{r^2} + D$$

- Stress component

$$\sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = -(2A + \frac{6C}{r^4} + \frac{4D}{r^2})\cos 2\theta$$

$$\sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2} = (2A + 12Br^2 + \frac{6C}{r^4})\cos 2\theta$$

$$\sigma_{r\theta} = \frac{\partial\phi}{\partial r}(\frac{1}{r}\frac{\partial\phi}{\partial\theta}) = (2A + 6Br^2 - \frac{6C}{r^4} - \frac{2D}{r^2})\sin 2\theta$$

- Applying BCs

$$2A + \frac{6C}{b^4} + \frac{4D}{b^2} = -\frac{S}{2}$$

$$2A + \frac{6C}{a^4} + \frac{4D}{a^2} = 0$$

$$2A + 6Bb^2 - \frac{6C}{b^4} - \frac{2D}{b^2} = -\frac{S}{2}$$

$$2A + 6Ba^2 - \frac{6C}{a^4} - \frac{2D}{a^2} = 0$$

- As $b \rightarrow \infty$

$$A = -\frac{S}{4} \;,\quad B = 0 \;,\quad C = -\frac{a^4}{4}S \;,\quad D = \frac{a^2}{2}S$$

- Final Solution

$$\sigma_{rr} = \frac{S}{2}(1-\frac{a^2}{r^2}) + \frac{S}{2}(1+\frac{3a^4}{r^4}-\frac{4a^2}{r^2})\cos 2\theta$$

$$\sigma_{\theta\theta} = \frac{S}{2}(1+\frac{a^2}{r^2}) - \frac{S}{2}(1+\frac{3a^4}{r^4})\cos 2\theta$$

$$\sigma_{r\theta} = -\frac{S}{2}(1-\frac{3a^4}{r^4}+\frac{2a^2}{r^2})\sin 2\theta$$

- Stress concentration on $r = a$

$$\sigma_{rr} = \sigma_{r\theta} = 0$$

$$\sigma_{\theta\theta} = S - 2S\cos 2\theta$$

– at $\theta = \frac{1}{2}\pi$, $\frac{3}{2}\pi$: $\sigma_{\theta\theta} = 3S$

– at $\theta = 0$, π : $\sigma_{\theta\theta} = -S$ (Compression)

– at $\theta = \frac{1}{2}\pi$: $\sigma_{\theta\theta} = S(1+\frac{1}{2}\frac{a^2}{r^2}+\frac{3}{2}\frac{a^4}{r^4})$ (rapidly decaying)

- Tensile traction in both *x*- and *y*- direction

$$\sigma_{rr} = \frac{S}{2}(1-\frac{a^2}{r^2}) + \frac{S}{2}(1+\frac{3a^4}{r^4}-\frac{4a^2}{r^2})\cos 2\theta + \frac{S}{2}(1-\frac{a^2}{r^2}) + \frac{S}{2}(1+\frac{3a^4}{r^4}-\frac{4a^2}{r^2})\cos(2\theta+\pi)$$

$$= S(1-\frac{a^2}{r^2})$$

$$\sigma_{\theta\theta} = S(1+\frac{a^2}{r^2})$$

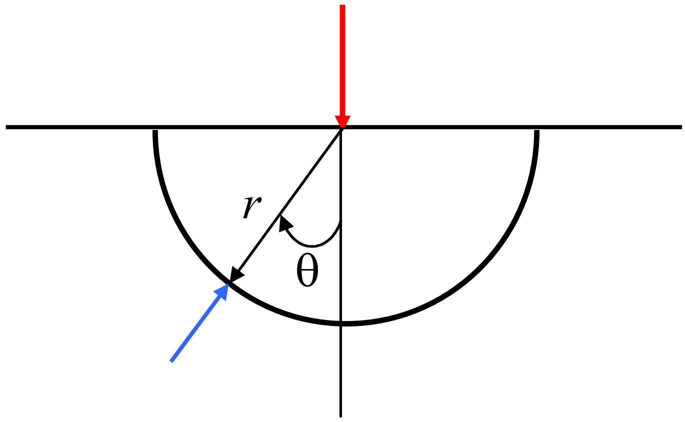
$$\sigma_{r\theta} = 0$$

- Opposite tractions (pure shear case)

$$\sigma_{\theta\theta} = \frac{S}{2}(1+\frac{a^2}{r^2}) - \frac{S}{2}(1+\frac{3a^4}{r^4})\cos 2\theta - \frac{S}{2}(1+\frac{a^2}{r^2}) + \frac{S}{2}(1+\frac{3a^4}{r^4})\cos(2\theta+\pi) \quad \rightarrow (\sigma_{\theta\theta})_{\max} = \pm 4S$$

$$= -S(1+\frac{3a^2}{r^4})\cos 2\theta$$

6.9. Flamant solution (2-D Boussinesq Solution)



- Boundary condition

$$\sigma_{\theta\theta} = \sigma_{r\theta} = 0 \quad \text{on} \quad \theta = \pm\frac{\pi}{2}$$

- Equilibrium condition

$$\int_{-\pi/2}^{\pi/2} \sigma_{rr} \cos\theta r d\theta = -P \quad \text{for all } r. \rightarrow \sigma_{rr} = \frac{f(\theta)}{r}$$

$$\sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = \frac{f(\theta)}{r}, \quad \sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2} = 0$$

$$(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2})(\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}) = (\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2})\frac{f(\theta)}{r} = 0$$

$$f(\theta) + f''(\theta) = 0 \rightarrow f(\theta) = A\cos\theta + B\sin\theta$$

By symmetry of stress, $B = 0$

$$\int_{-\pi/2}^{\pi/2} A\cos^2\theta d\theta = \frac{1}{2}\int_{-\pi/2}^{\pi/2} (A(1+\cos 2\theta) d\theta = \frac{\pi}{2} A = -P$$

- Stress components (Polar)

$$\sigma_{rr} = -\frac{2P}{\pi}\frac{\cos\theta}{r}, \quad \sigma_{\theta\theta} = \sigma_{r\theta} = 0$$

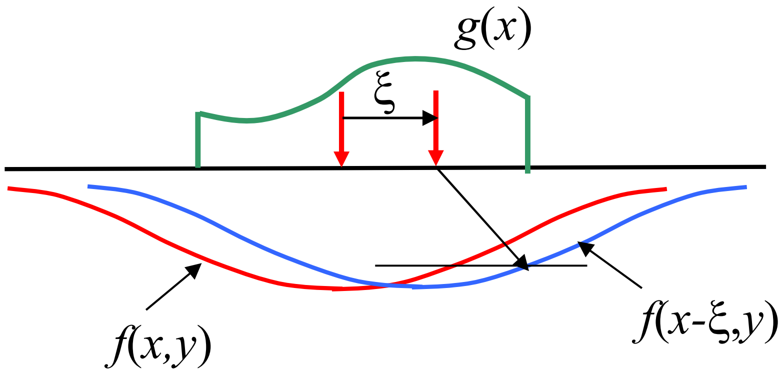
- Stress components (Cartesian)

$$\sigma_{xx} = \sigma_{rr}\cos^2\theta = -\frac{2P}{\pi r}\cos^3\theta = -\frac{2P}{\pi a}\cos^4\theta$$

$$\sigma_{yy} = \sigma_{rr}\sin^2\theta = -\frac{2P}{\pi a}\cos^2\theta\sin^2\theta$$

$$\sigma_{xy} = \sigma_{rr}\sin\theta\cos\theta = -\frac{2P}{\pi a}\sin\theta\cos^3\theta$$

- Convolution Integral (Influence line)



$$dR = f(x-\xi, y)g(\xi)d\xi \rightarrow R = \int_a^b f(x-\xi, y)g(\xi)d\xi$$

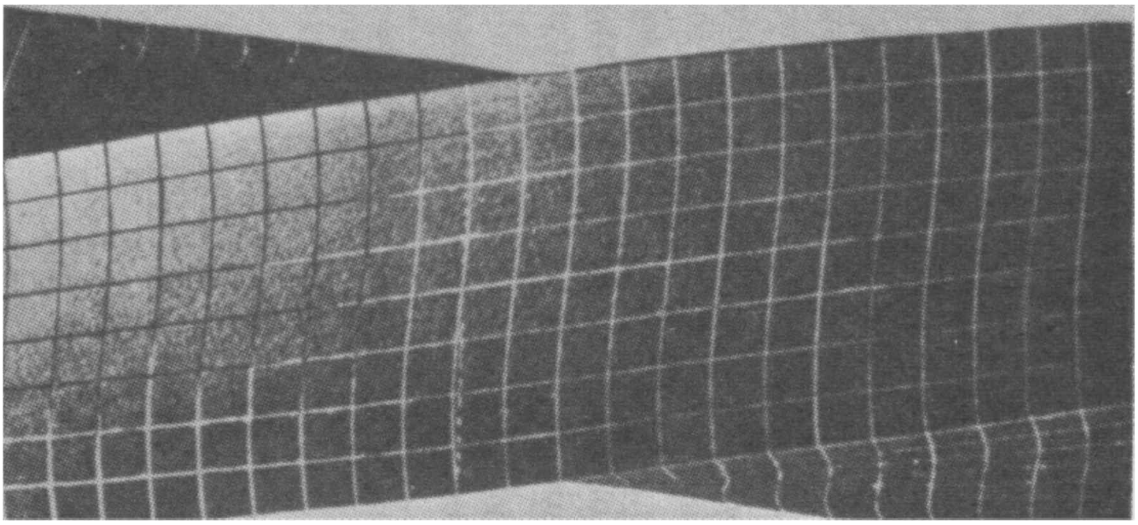
– Moment Case

$$R = \lim_{\varepsilon \to 0}(Pf(x,y) - Pf(x-\varepsilon, y)) = \lim_{\varepsilon \to 0}(P\varepsilon \frac{f(x,y) - f(x-\varepsilon, y)}{\varepsilon}) =$$

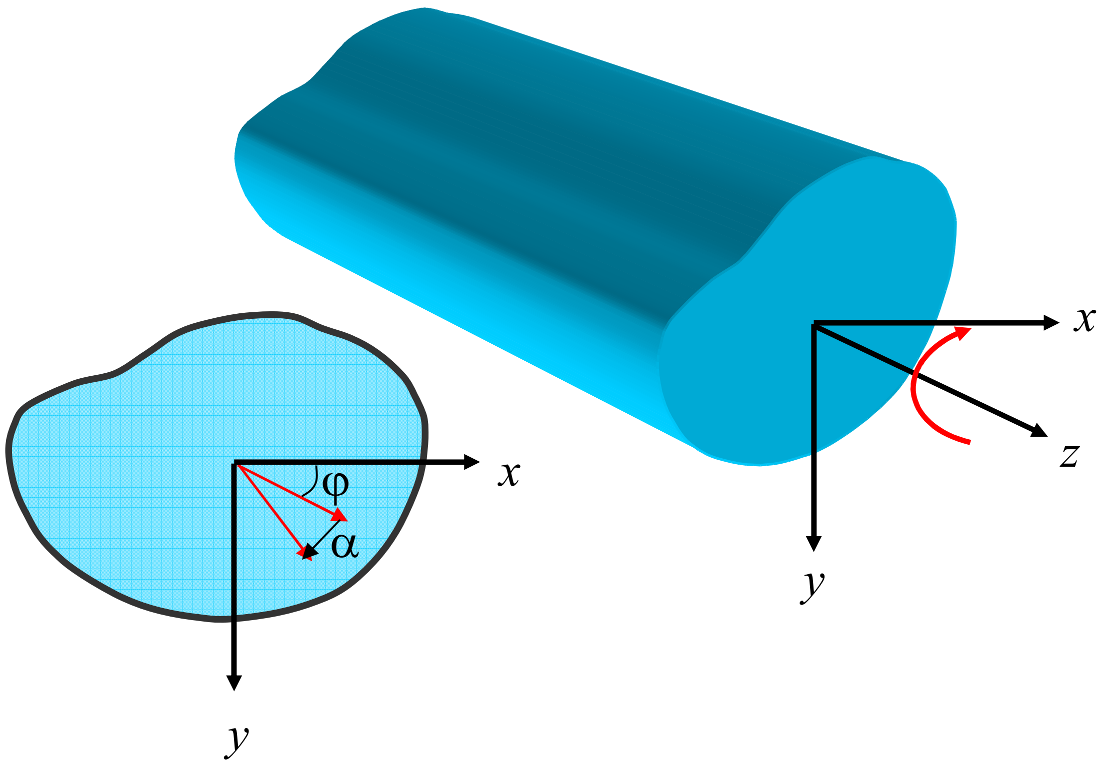
$$= M \lim_{\varepsilon \to 0}(\frac{f(x,y) - f(x-\varepsilon, y)}{\varepsilon}) = M \frac{\partial f}{\partial x}$$

Chapter 7

Uniform Torsion (St. Venant Torsion)



7.1. Basics



- Displacement

$$u_x = r(\cos(\alpha+\varphi)-\cos\varphi) = r(\cos\alpha\cos\varphi-\sin\alpha\sin\varphi-\cos\varphi)$$

$$\approx -r\sin\alpha\sin\varphi \approx -r\alpha\frac{y}{r} = -\alpha y$$

$$u_y = r(\sin(\alpha+\varphi)-\sin\varphi) = r(\sin\alpha\cos\varphi+\cos\alpha\sin\varphi-\sin\varphi) \approx r\sin\alpha\cos\varphi \approx r\alpha\frac{x}{r} = \alpha x$$

$\alpha = \theta z$, θ : rotation angle per unit length

- Warping function

$$u_z = \theta\psi(x,y)$$

- Strain components

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} = -\frac{\partial \theta z y}{\partial x} = 0 \qquad \gamma_{xy} = \frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x} = -\frac{\partial \theta z y}{\partial y}+\frac{\partial \theta z x}{\partial x} = 0$$

$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y} = \frac{\partial \theta z x}{\partial y} = 0 \qquad \gamma_{xz} = \frac{\partial u_x}{\partial z}+\frac{\partial u_z}{\partial x} = -\frac{\partial \theta z y}{\partial z}+\frac{\partial \theta\psi}{\partial x} = \theta(\frac{\partial\psi}{\partial x}-y)$$

$$\varepsilon_{zz} = \frac{\partial u_z}{\partial z} = -\frac{\partial \theta\psi(x,y)}{\partial z} = 0 \qquad \gamma_{yz} = \frac{\partial u_y}{\partial z}+\frac{\partial u_z}{\partial y} = \frac{\partial \theta z x}{\partial z}+\frac{\partial \theta\psi}{\partial y} = \theta(\frac{\partial\psi}{\partial y}+x)$$

- Stress components

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma_{xy} = 0$$
$$\sigma_{xz} = G\theta\left(\frac{\partial\psi}{\partial x} - y\right)$$
$$\sigma_{yz} = G\theta\left(\frac{\partial\psi}{\partial y} + x\right)$$

- Equilibrium equation (z-direction only)

$$\frac{\partial\sigma_{xz}}{\partial x} + \frac{\partial\sigma_{yz}}{\partial y} + \frac{\partial\sigma_{zz}}{\partial z} = G\theta\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) = 0 \rightarrow \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0$$

- Traction free BC on the longitudinal surface

$$T_i = \sigma_{ij} n_j = 0 \quad \text{for all } i, \quad \mathbf{n} = (n_x, n_y, 0)$$
$$T_x = \sigma_{xx} n_x + \sigma_{xy} n_y = 0$$
$$T_y = \sigma_{yx} n_x + \sigma_{yy} n_y = 0$$
$$T_z = \sigma_{zx} n_x + \sigma_{zy} n_y = 0$$

The first two BCs are automatically satisfied, and

$$\sigma_{zx} n_x + \sigma_{zy} n_y = \sigma_{zx}\frac{dy}{ds} - \sigma_{zy}\frac{dx}{ds} = 0$$
$$\left(\frac{\partial\psi}{\partial x} - y\right)\frac{dy}{ds} - \left(\frac{\partial\psi}{\partial y} + x\right)\frac{dx}{ds} = 0$$

- Stress function

$$\sigma_{xz} = \frac{\partial\phi}{\partial y} = G\theta\left(\frac{\partial\psi}{\partial x} - y\right)$$
$$\sigma_{yz} = -\frac{\partial\phi}{\partial x} = G\theta\left(\frac{\partial\psi}{\partial y} + x\right)$$

by eliminating ψ,

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = -2G\theta$$

The traction BC becomes

$$\frac{\partial\phi}{\partial y}\frac{dy}{ds} + \frac{\partial\phi}{\partial x}\frac{dx}{ds} = \frac{d\phi}{ds} = 0 \rightarrow \phi \quad \text{is constant on the boundary.}$$

- BCs at the end of the bar

$$\int_A \sigma_{zx} dA = \int_A \frac{\partial \phi}{\partial y} dA = \int_S \phi n_y dS = 0$$

$$\int_A \sigma_{zy} dA = -\int_A \frac{\partial \phi}{\partial x} dA = -\int_S \phi n_x dS = 0$$

$$M_t = \int_A \sigma_{zy} x dA - \int_A \sigma_{zx} y dA = -\int_A \frac{\partial \phi}{\partial x} x dA - \int_A \frac{\partial \phi}{\partial y} y dA$$
$$= -\int_A \nabla\phi \cdot \mathbf{X} dA = -\int_S \phi \mathbf{X} \cdot \mathbf{n} dS + 2\int_A \phi dA = 2\int_A \phi dA$$

- Summary
  - Governing equation : $\dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} = -2G\theta$
  - Boundary condition : $\phi = 0$
  - Twisting moment : $M_t = 2\int_A \phi dA$

7.2. Torsion in an Elliptic Section

- Elliptic section

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$$

- Stress function

$$\phi = m(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1)$$

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=-2G\theta \rightarrow m(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2})(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1)=-2G\theta$$

$$m(\frac{2}{a^2}+\frac{2}{b^2})=-2G\theta \rightarrow m=-\frac{a^2b^2}{a^2+b^2}G\theta$$

$$\phi = -\frac{a^2b^2}{a^2+b^2}(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1)G\theta$$

- Twisting angle

$$M_t = -2G\theta\int_A \frac{a^2b^2}{a^2+b^2}(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1)dA$$

$$= -2G\theta\frac{a^2b^2}{a^2+b^2}(\frac{a^3b\pi}{4a^2}+\frac{ab^3\pi}{4b^2}-ab\pi) \rightarrow \theta = \frac{a^2+b^2}{a^3b^3\pi}\frac{M_t}{G}$$

$$= G\theta\frac{a^3b^3\pi}{a^2+b^2}$$

$$\phi = -\frac{M_t}{ab\pi}(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1)$$

- Stress

$$\sigma_{xz}=\frac{\partial\phi}{\partial y}=-\frac{2M_t}{ab^3\pi}y \;,\quad \sigma_{yz}=-\frac{\partial\phi}{\partial x}=\frac{2M_t}{a^3b\pi}x$$

- Warping function

$$-\frac{2M_t}{ab^3\pi}y = G\theta(\frac{\partial\psi}{\partial x}-y) \rightarrow -\frac{2M_t}{ab^3\pi}xy = G\theta(\psi - xy)+f(y)$$

$$\frac{2M_t}{a^3b\pi}x = G\theta(\frac{\partial\psi}{\partial y}+x) \rightarrow \quad \frac{2M_t}{a^3b\pi}xy = G\theta(\psi + xy)+g(x)$$

$$\frac{2M_t}{a^3b^3\pi}xy(b^2+a^2)=\frac{2M_t}{a^3b^3\pi}(b^2+a^2)xy+g(x)-f(y)$$

$$g(x)=f(y)=c$$

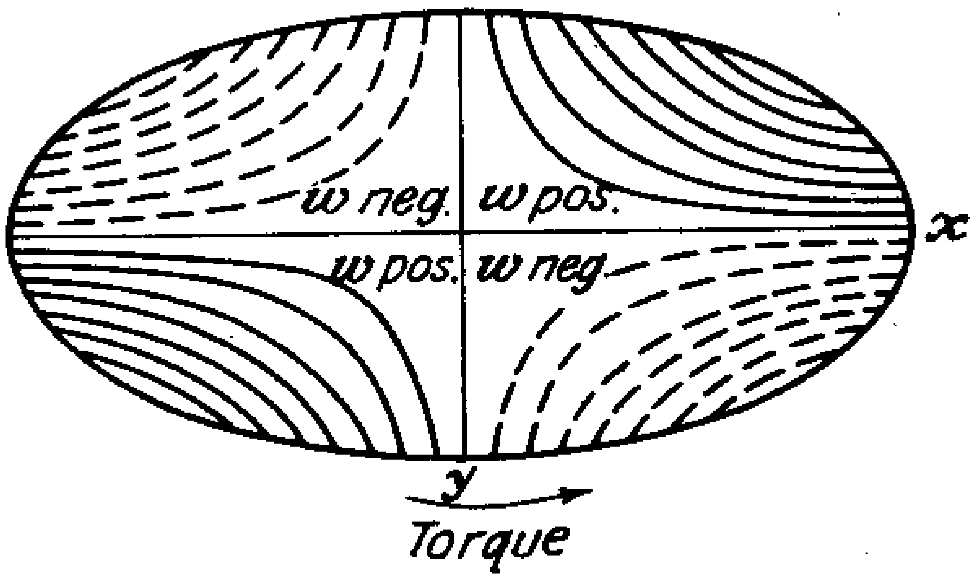
$$\frac{2M_t}{a^3b^3\pi}xy(b^2-a^2)=\frac{2M_t}{a^3b^3\pi}(b^2+a^2)\psi+2c$$

$$\psi=\frac{b^2-a^2}{b^2+a^2}xy-\frac{c}{M_t}\frac{a^3b^3\pi}{b^2+a^2}$$

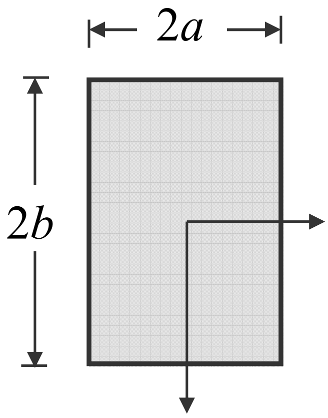
$$\psi(0,0)=0\rightarrow c=0$$

$$\psi=\frac{b^2-a^2}{b^2+a^2}xy$$

For a circular section, the warping does not occur



7.3. Rectangular Section (Series Solution)



- Governing equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -2G\theta$$

- BCs: $\phi = 0$ on $x = \pm a$ and $y = \pm b$

- Series expansion

$$\phi = \sum_{n=1,3,5,\ldots}^{\infty} \cos\frac{n\pi x}{2a} Y_n \quad \text{(BCs on } x = \pm a \text{ are satisfied.)}$$

$$-2G\theta = -\sum_{n=1,3,5,\ldots}^{\infty} 2G\theta \frac{4}{n\pi}(-1)^{(n-1)/2} \cos\frac{n\pi x}{2a}$$

- ODE in y-direction

$$\sum_{n=1,3,5,\ldots}^{\infty} \cos\frac{n\pi x}{2a}\left(-\left(\frac{n\pi}{2a}\right)^2 Y_n + Y_n'' + G\theta\frac{8}{n\pi}(-1)^{(n-1)/2}\right) = 0$$

$$Y_n'' - \left(\frac{n\pi}{2a}\right)^2 Y_n = -G\theta\frac{8}{n\pi}(-1)^{(n-1)/2} \quad \text{for } n = 1,3,5\ldots$$

$$Y_n = A\sinh\frac{n\pi y}{2a} + B\cosh\frac{n\pi y}{2a} + G\theta\frac{32a^2}{(n\pi)^3}(-1)^{(n-1)/2}$$

- Application BCs on $y = \pm b$

$$(Y_n)_{y=\pm b} = \pm A\sinh\frac{n\pi b}{2a} + B\cosh\frac{n\pi b}{2a} + G\theta\frac{32a^2}{(n\pi)^3}(-1)^{(n-1)/2} = 0$$

$$A = 0, \quad B = -G\theta\frac{32a^2}{(n\pi)^3}\frac{1}{\cosh(n\pi b/2a)}(-1)^{(n-1)/2}$$

$$Y_n = G\theta\frac{32a^2}{(n\pi)^3}(-1)^{(n-1)/2}\left(1 - \frac{\cosh(n\pi y/2a)}{\cosh(n\pi b/2a)}\right)$$

- Final Solution

$$\phi = \frac{32G\theta a^2}{\pi^3}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^3}(-1)^{(n-1)/2}\cos\frac{n\pi x}{2a}\left(1-\frac{\cosh(n\pi y/2a)}{\cosh(n\pi b/2a)}\right)$$

- Stress

$$\sigma_{yz} = -\frac{\partial\phi}{\partial x} = \frac{16G\theta a}{\pi^2}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^2}(-1)^{(n-1)/2}\sin\frac{n\pi x}{2a}\left(1-\frac{\cosh(n\pi y/2a)}{\cosh(n\pi b/2a)}\right)$$

$$\sigma_{xz} = \frac{\partial\phi}{\partial y} = -\frac{16G\theta a}{\pi^2}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^2}(-1)^{(n-1)/2}\cos\frac{n\pi x}{2a}\frac{\sinh(n\pi y/2a)}{\cosh(n\pi b/2a)}$$

- Maximum Stress

$$\sigma_{\max} = (\sigma_{yz})_{(x=a,y=0)} = \frac{16G\theta a}{\pi^2}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^2}(-1)^{(n-1)/2}\sin\frac{n\pi}{2}\left(1-\frac{1}{\cosh(n\pi b/2a)}\right)$$

$$= \frac{16G\theta a}{\pi^2}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^2}\left(1-\frac{1}{\cosh(n\pi b/2a)}\right)$$

$$= 2G\theta a - \frac{16G\theta a}{\pi^2}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^2}\frac{1}{\cosh(n\pi b/2a)} \quad , \left(1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots=\frac{\pi^2}{8}\right)$$

- Twisting angle

$$M_t = 2\int_A \phi\, dA$$

$$= \frac{64G\theta a^2}{\pi^3}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^3}(-1)^{(n-1)/2}\int_{-a}^{a}\cos\frac{n\pi x}{2a}dx\int_{-b}^{b}\left(1-\frac{\cosh(n\pi y/2a)}{\cosh(n\pi b/2a)}\right)dy$$

$$= \frac{64G\theta a^2}{\pi^3}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^3}2\frac{2a}{n\pi}2\left(b-\frac{2a}{n\pi}\frac{\sinh(n\pi b/2a)}{\cosh(n\pi b/2a)}\right)$$

$$= \frac{32G\theta(2a)^3 2b}{\pi^4}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^4} - \frac{64G\theta(2a)^4}{\pi^5}\sum_{n=1,3,5,\ldots}^{\infty}\frac{\tanh(n\pi b/2a)}{n^5}$$

$$= \frac{G\theta(2a)^3 2b}{3}\left(1-\frac{192}{\pi^5}\frac{a}{b}\sum_{n=1,3,5,\ldots}^{\infty}\frac{\tanh(n\pi b/2a)}{n^5}\right) \quad , \left(1+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\cdots=\frac{\pi^4}{96}\right)$$

- Narrow section $\frac{b}{a} \geq 3$

$$\tanh(\pi b/2a) = \frac{111.3178-0.0090}{111.3178+0.0090} = 0.9998 \approx 1$$

$$M_t = \frac{G\theta(2a)^3 2b}{3}\left(1-0.6274\frac{a}{b}\left(1+\frac{1}{243}+\frac{1}{3125}+\cdots\right)\right) \approx \frac{G\theta(2a)^3 2b}{3}\left(1-0.6274\frac{a}{b}\right)$$

- Square section

$$\tanh(\pi/2) = \frac{4.8105 - 0.2079}{4.8105 + 0.2079} = 0.9171$$

$$\begin{aligned} M_t &= \frac{G\theta(2a)^4}{3}\left(1 - 0.6274\left(0.9171 + \frac{1}{243} + \frac{1}{3125} + \cdots\right)\right) \\ &\approx \frac{G\theta(2a)^4}{3}(1 - 0.6274 \times 0.9171) \\ &= 0.1415 G\theta(2a)^4 \end{aligned}$$

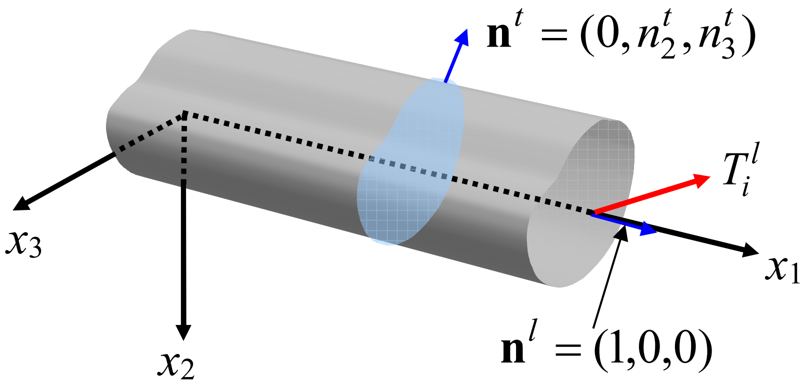
- Open section with constant thickness

$$M_t \approx \frac{G\theta}{3}\sum_i w_i t_i^3$$

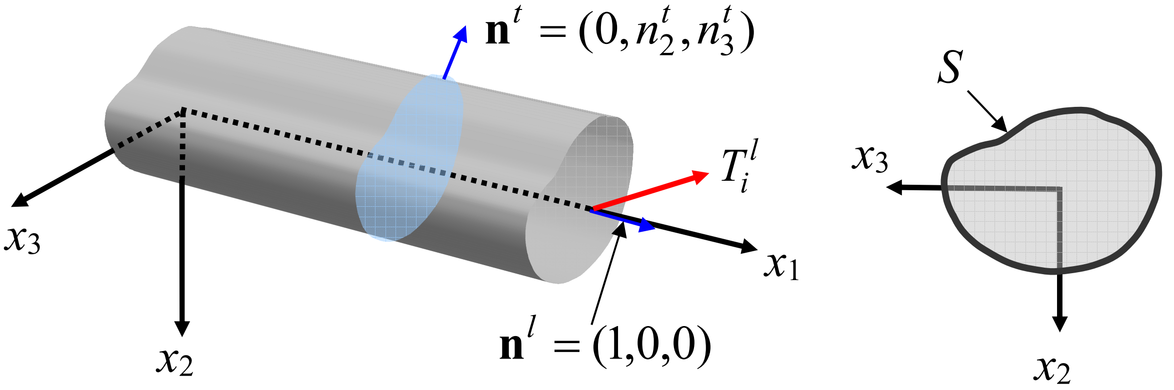
Chapter 8

Beam Approximation

$\cong$



8.1. Equilibrium Equation



- Force Resultant

$$Q_i = \int_A T_i^l dA = \int_A \sigma_{ji} n_j^l dA = \int_A \sigma_{1i} dA$$

$$\frac{\partial Q_i}{\partial x_1} = \int_A \frac{\partial \sigma_{1i}}{\partial x_1} dA = \int_A \left(\frac{\partial \sigma_{ji}}{\partial x_j} - \frac{\partial \sigma_{2i}}{\partial x_2} - \frac{\partial \sigma_{3i}}{\partial x_3}\right) dA$$

$$= \int_A \left(\frac{\partial \sigma_{ji}}{\partial x_j} + b_i\right) dA - \int_A b_i dA - \int_A \left(\frac{\partial \sigma_{2i}}{\partial x_2} + \frac{\partial \sigma_{3i}}{\partial x_3}\right) dA$$

$$= \int_A \left(\frac{\partial \sigma_{ji}}{\partial x_j} + b_i\right) dA - \int_A b_i dA - \int_S (\sigma_{2i} n_2^t + \sigma_{3i} n_3^t) dS$$

$$= \int_A \left(\frac{\partial \sigma_{ji}}{\partial x_j} + b_i\right) dA - \int_A b_i dA - \int_S T_i^t dS = \int_A \left(\frac{\partial \sigma_{ji}}{\partial x_j} + b_i\right) dA - q_i$$

$$\frac{\partial Q_i}{\partial x_1} + q_i = \int_A \left(\frac{\partial \sigma_{ji}}{\partial x_j} + b_i\right) dA = 0$$

- Moment Resultant

$$\tilde M_i = \int_A \tilde{\mathbf{x}} \times \mathbf{T}^l dA = \int_A e_{ijk}\tilde x_j T_k^l dA = \int_A e_{ijk}\tilde x_j \sigma_{km} n_m^l dA = \int_A e_{ijk}\tilde x_j \sigma_{k1} dA \qquad \text{where} \quad \tilde{\mathbf{x}} = (0, x_2, x_3)$$

$$\frac{\partial \tilde M_i}{\partial x_1} = \int_A e_{ijk}\tilde x_j \frac{\partial \sigma_{k1}}{\partial x_1} dA$$

$$= \int_A e_{ijk}\tilde x_j \left(\frac{\partial \sigma_{km}}{\partial x_m} - \frac{\partial \sigma_{k2}}{\partial x_2} - \frac{\partial \sigma_{k3}}{\partial x_3}\right) dA$$

$$= \int_A e_{ijk}\tilde x_j \frac{\partial \sigma_{km}}{\partial x_m} dA - \int_A e_{ijk}\tilde x_j \left(\frac{\partial \sigma_{k2}}{\partial x_2} + \frac{\partial \sigma_{k3}}{\partial x_3}\right) dA$$

$$= \int_A e_{ijk}\tilde x_j \left(\frac{\partial \sigma_{km}}{\partial x_m} + b_k\right) dA - \int_A e_{ijk}\tilde x_j b_k dA - \int_A \left(\frac{\partial e_{ijk}\tilde x_j \sigma_{k2}}{\partial x_2} + \frac{\partial e_{ijk}\tilde x_j \sigma_{k3}}{\partial x_3}\right) dA + \int_A (e_{i2k}\sigma_{k2} + e_{i3k}\sigma_{k3}) dA$$

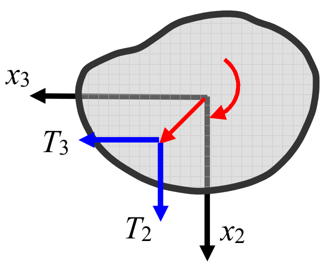
$$= \int_A e_{ijk}\tilde x_j \left(\frac{\partial \sigma_{km}}{\partial x_m} + b_k\right) dA - \int_A e_{ijk}\tilde x_j b_k dA - \int_S e_{ijk}\tilde x_j \sigma_{km} n_m^t dS + \int_A (e_{i2k}\sigma_{k2} + e_{i3k}\sigma_{k3}) dA$$

$$= \int_A e_{ijk}\tilde x_j \left(\frac{\partial \sigma_{km}}{\partial x_m} + b_k\right) dA - \int_A e_{ijk}\tilde x_j b_k dA - \int_S e_{ijk}\tilde x_j T_k^t dS - e_{ijk} n_j^l Q_k$$

$$\underline{\underline{\frac{\partial \tilde M_i}{\partial x_1} + \int_A e_{ijk}\tilde x_j b_k dA + \int_S e_{ijk}\tilde x_j T_k^t dS + e_{ijk} n_j^l Q_k = 0}}$$

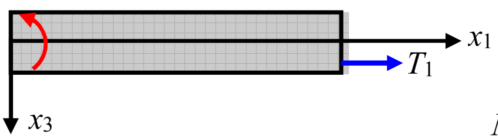
- Sign Convention for Moment

For i =1



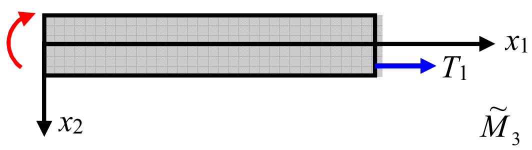
$$\tilde M_1 = \int_A e_{1jk}\tilde x_j T_k^l dA = \int_A (e_{123}\tilde x_2 T_3^l + e_{132}\tilde x_3 T_2^l) dA = \int_A (x_2 T_3^l - x_3 T_2^l) dA = M_1$$

For i =2



$$\tilde M_2 = \int_A e_{2jk}\tilde x_j T_k^l dA = \int_A e_{231}\tilde x_3 T_1^l dA = \int_A x_3 T_1^l dA = M_2$$

For i =3



$$\tilde M_3 = \int_A e_{3jk}\tilde x_j T_k^l dA = \int_A e_{321}\tilde x_2 T_1^l dA = -\int_A x_3 T_1^l dA = -M_3$$

- Component Form

$$\frac{\partial Q_1}{\partial x_1} + q_1 = 0 \qquad q_1 = \int_A b_1 dA + \int_S T_1^t dS$$

$$\frac{\partial Q_2}{\partial x_1} + q_2 = 0 \qquad q_2 = \int_A b_2 dA + \int_S T_2^t dS$$

$$\frac{\partial Q_3}{\partial x_1} + q_3 = 0 \quad q_3 = \int_A b_3 dA + \int_S T_3^t dS$$

$$\frac{\partial M_1}{\partial x_1} + m_1 = 0 \qquad m_1 = \int_A (x_2 b_3 - x_3 b_2) dA + \int_S (\tilde{x}_2 T_3^t - x_3 T_2^t) dS$$

$$\frac{\partial M_2}{\partial x_1} + m_2 - Q_3 = 0 \qquad m_2 = \int_A x_3 b_1 dA + \int_S x_3 T_1^t dS$$

$$\frac{\partial M_3}{\partial x_1} - m_3 - Q_2 = 0 \qquad m_3 = \int_A x_2 b_1 dA + \int_S x_2 T_1^t dS$$

- Planar Beam $b_1 = b_3 = T_1^t = T_3^t = 0$

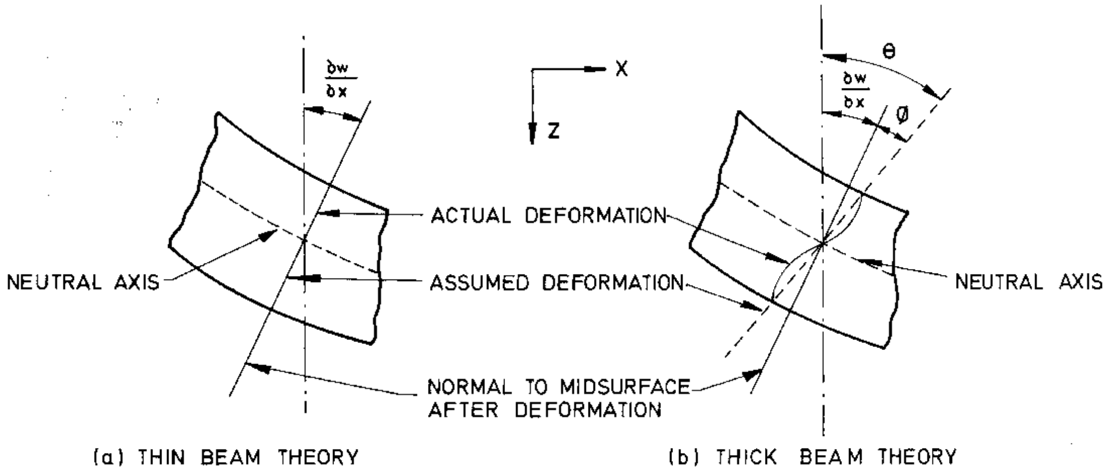
$$\frac{\partial Q_2}{\partial x_1} + q_2 = 0 \quad , \quad \frac{\partial M_3}{\partial x_1} - Q_2 = 0 \;\longrightarrow\; \frac{\partial^2 M_3}{\partial x_1^2} = -q_2$$

$$\frac{\partial M_1}{\partial x_1} + m_1 = 0 \qquad m_1 = -\int_A x_3 b_2 dA - \int_S x_3 T_2^t dS$$

What condition(s) should be satisfied in order that the vertical loads do not cause torsion of beam ??

8.2. Bernoulli Beams

- Assumption on displacement field
  - A plane section remains plane after deformation.
  - Normal remains normal.
  - The vertical displacement is constant through the depth of a beam.



- Displacement Components by assumption

$$u_x = -\frac{\partial u_y}{\partial x} y \quad , \quad u_y = u_y(x) \quad , \quad u_z = ??$$

- Strain-displacement relation

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} = -\frac{\partial^2 u_y}{\partial x^2} y \qquad \gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = 0$$

$$\varepsilon_{yy} = 0 \qquad \gamma_{yz} = \frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z} = \frac{\partial u_z}{\partial y} = ??$$

$$\varepsilon_{zz} = \frac{\partial u_z}{\partial z} = ?? \qquad \gamma_{xz} = \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} = \frac{\partial u_z}{\partial x} = ??$$

- Strain-Stress relation

$$\varepsilon_{11} = \frac{\sigma_{11}}{E} - \nu\left(\frac{\sigma_{22}}{E} + \frac{\sigma_{33}}{E}\right)$$

$$\varepsilon_{22} = \frac{\sigma_{22}}{E} - \nu\left(\frac{\sigma_{11}}{E} + \frac{\sigma_{33}}{E}\right)$$

$$\varepsilon_{33} = \frac{\sigma_{33}}{E} - \nu\left(\frac{\sigma_{11}}{E} + \frac{\sigma_{22}}{E}\right)$$

- Assumption on stress

  – By traction BC, $\sigma_{zz} = 0$, $\sigma_{yy} = 0$ (??)

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E}, \ \varepsilon_{yy} = -\nu\frac{\sigma_{xx}}{E}, \ \varepsilon_{zz} = -\nu\frac{\sigma_{zz}}{E}$$

Since, however, $\varepsilon_{yy} = 0$ by the strain-displacement relation, the Poisson ratio should be zero, which is equivalent to neglecting the Poisson effect.

$$\sigma_{xx} = -E\frac{\partial^2 u_y}{\partial x^2}y$$

$$M_z = \int_A \sigma_{xx} y dA = -\int_A E\frac{\partial^2 u_y}{\partial x^2}y^2 dA = -E\frac{\partial^2 u_y}{\partial x^2}\int_A y^2 dA = -EI\frac{\partial^2 u_y}{\partial x^2}$$

- Equilibrium Equation

$$\begin{array}{l}\dfrac{\partial\sigma_{xx}}{\partial x} + \dfrac{\partial\sigma_{xy}}{\partial y} + b_x = 0 \\ \dfrac{\partial\sigma_{xy}}{\partial x} + \dfrac{\partial\sigma_{yy}}{\partial y} + b_y = 0\end{array} \rightarrow \begin{array}{l}\dfrac{\partial\sigma_{xx}}{\partial x} + \dfrac{\partial\sigma_{xy}}{\partial y} = 0 \\ \dfrac{\partial\sigma_{xy}}{\partial x} + b_y = 0\end{array}$$

$$\int_A \left(\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y}\right) y dA = 0 \rightarrow \int_A \left(\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y}\right) y dA = 0 \rightarrow \frac{\partial}{\partial x}\int_A \sigma_{xx} y dA + \int_A \frac{\partial\sigma_{xy}}{\partial y} y dA = 0 \rightarrow$$

$$\frac{\partial M}{\partial x} + \int_A \frac{\partial\sigma_{xy}}{\partial y} y dA = 0 \rightarrow \frac{\partial M}{\partial x} + \int_z \sigma_{xy} y\Big|_{h_1}^{h_2} dz - \int_A \sigma_{xy} dA = 0 \rightarrow \frac{\partial M}{\partial x} - V = 0$$

$$\int_A \left(\frac{\partial\sigma_{xy}}{\partial x} + b_y\right) dA = 0 \rightarrow \frac{\partial}{\partial x}\int_A \sigma_{xy} dA + \int_A b_y dA = 0 \rightarrow \frac{\partial V}{\partial x} + q = 0$$

$$\frac{\partial^2 M}{\partial x^2} + q = 0$$

$$\frac{\partial^2 M}{\partial x^2} = -q \rightarrow \frac{\partial^2}{\partial x^2} EI\frac{\partial^2 u_y}{\partial x^2} = q \quad , \quad \sigma_{xx} = \frac{M}{I}y$$

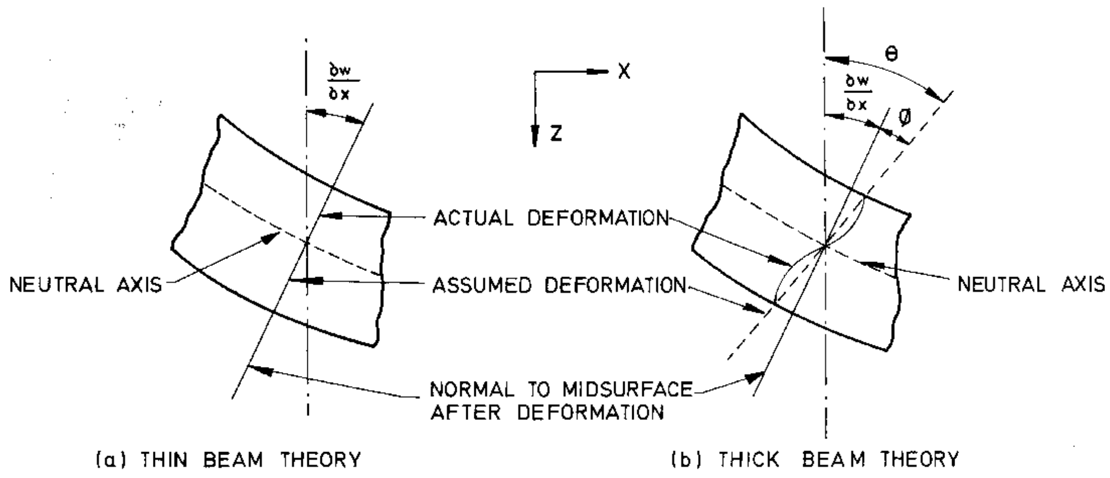
$$\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} = 0 \rightarrow \int_y^{h_2} \left(\frac{\partial M}{\partial x}\frac{y}{I} + \frac{\partial\sigma_{xy}}{\partial y}\right) dA = 0 \rightarrow \frac{V}{I}\int_y^{h_2} y dA + \int_y^{h_2} \frac{\partial\sigma_{xy}}{\partial y} b dy = 0 \rightarrow \sigma_{xy}(y) - \sigma_{xy}(h_2) = \frac{VQ}{Ib}$$

- Simply supported beam with a uniformly distributed load

$$(\sigma_{xx})_{\max} = \frac{6}{8}\frac{ql^2}{h^2} \ , \ (\sigma_{yy})_{\max} \approx q \quad \frac{(\sigma_{yy})_{\max}}{(\sigma_{xx})_{\max}} \approx \frac{8}{6}\frac{h^2}{l^2}$$

5.3. Timoshenko Beams

- Assumption on displacement field
  - A plane section remains plane after deformation.
  - ~~Normal remains normal.~~
  - The vertical displacement is constant through the depth of a beam.



- Displacement Components by assumption

$$u_x = -\theta(x) y \quad , \quad u_y = u_y(x) \quad , \quad u_z = ??$$

- Strain Components

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} = -\frac{\partial \theta}{\partial x} y \qquad \gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = -\theta + \frac{\partial u_y}{\partial x}$$

$$\varepsilon_{yy} = 0 \qquad \gamma_{yz} = \frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z} = \frac{\partial u_z}{\partial y} = ??$$

$$\varepsilon_{zz} = \frac{\partial u_z}{\partial z} = ?? \qquad \gamma_{xz} = \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} = \frac{\partial u_z}{\partial x} = ??$$

- Equilibrium equation

$$\frac{\partial M}{\partial x} = V \quad , \quad \frac{\partial V}{\partial x} + q = 0$$

- By energy consideration

$$\int_A \sigma_{xy}\gamma_{xy} dA = \gamma_{xy}\int_A \sigma_{xy} dA = \gamma_{xy} V = \int_A \sigma_{xy}\frac{\sigma_{xy}}{G}dA = \frac{V^2}{GI^2}\int_A \frac{Q^2}{b^2(y)}dA = \frac{V^2 f_s}{GA} = \frac{V^2}{GA_0}$$

which yield $V = GA_0\gamma_{xy}$

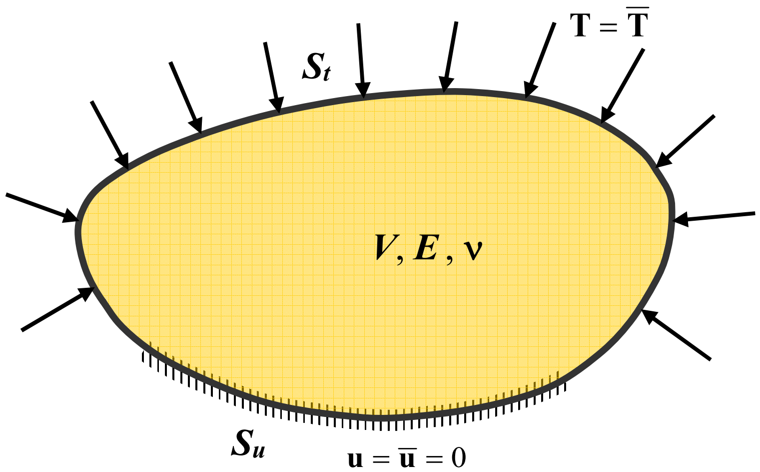
$$M = \int_A \sigma_{xx} y dA = -\int_A E\frac{\partial\theta}{\partial x}y^2 dA = -EI\frac{\partial\theta}{\partial x}$$

$$V = GA_0\left(\frac{\partial u_y}{\partial x} - \theta\right)$$

$$\left.\begin{aligned} &\frac{\partial M}{\partial x} = V \rightarrow -EI\frac{\partial^2\theta}{\partial x^2} = GA_0\left(\frac{\partial u_y}{\partial x} - \theta\right) \rightarrow EI\frac{\partial^2\theta}{\partial x^2} + GA_0\left(\frac{\partial u_y}{\partial x} - \theta\right) = 0 \\ &\frac{\partial V}{\partial x} + q = 0 \rightarrow GA_0\left(\frac{\partial^2 u_y}{\partial x^2} - \frac{\partial\theta}{\partial x}\right) = -q \end{aligned}\right\} \rightarrow EI\frac{\partial^3\theta}{\partial x^3} = q$$

Chapter 9
Energy Methods

9.1. Problem Definition



- Governing Equations and Boundary Conditions

Equilibrium Equation	: $\nabla \cdot \boldsymbol{\sigma} + \mathbf{b} = 0$	in V
Constitutive Law	: $\boldsymbol{\sigma} = \mathbf{D} : \boldsymbol{\varepsilon}$	in V
Strain-Displacement Relationship	: $\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T)$	in V
Displacement Boundary condition	: $\mathbf{u} - \overline{\mathbf{u}} = 0$	on S_u
Traction Boundary Condition	: $\mathbf{T} - \overline{\mathbf{T}} = 0$	on S_t
Cauchy's Relation on the Boundary	: $\mathbf{T} = \boldsymbol{\sigma} \cdot \mathbf{n}$	on S

- Energy Conservation

$$\begin{aligned}
\Pi_{ext} &= \frac{1}{2}\int_S u_i T_i dS + \frac{1}{2}\int_V u_i b_i dV \\
&= \frac{1}{2}\int_S u_i T_i dS - \frac{1}{2}\int_V u_i \frac{\partial \sigma_{ij}}{\partial x_j} dV \\
&= \frac{1}{2}\int_S u_i T_i dS + \frac{1}{2}\int_V \frac{\partial u_i}{\partial x_j}\sigma_{ij} dV - \frac{1}{2}\int_S u_i \sigma_{ij} n_j dS \\
&= \frac{1}{2}\int_S u_i T_i dS + \frac{1}{2}\int_V \frac{\partial u_i}{\partial x_j}\sigma_{ij} dV - \frac{1}{2}\int_S u_i T_i dS \\
&= \frac{1}{2}\int_V \frac{\partial u_i}{\partial x_j}\sigma_{ij} dV = \frac{1}{2}\int_V \varepsilon_{ij}\sigma_{ij} dV = \Pi_{\text{int}}
\end{aligned}$$

- For physical interpretation of the strain energy, please refer to pp. 244 - 246 of Theory of Elasticity by Timoshenko

- Total Potential Energy

$$\Pi = \frac{1}{2}\int_V \varepsilon_{ij}\sigma_{ij}\,dV - \int_V u_i\, b_i\,dV - \int_{\Gamma_t} u_i\,\overline{T}_i\,d\Gamma$$

9.2. Principle of Minimum Potential Energy

$$u_i^h = u_i - u_i^e$$

$$\Pi^h = \frac{1}{2}\int_V \varepsilon_{ij}^h \sigma_{ij}^h dV - \int_V u_i^h b_i dV - \int_{S_t} u_i^h \overline{T}_i dS$$

$$= \frac{1}{2}\int_V \frac{\partial(u_i - u_i^e)}{\partial x_j} D_{ijkl} \frac{\partial(u_k - u_k^e)}{\partial x_l} dV - \int_V (u_i - u_i^e) b_i dV - \int_{S_t} (u_i - u_i^e)\overline{T}_i dS$$

$$= \frac{1}{2}\int_V \frac{\partial u_i}{\partial x_j} D_{ijkl} \frac{\partial u_k}{\partial x_l} dV - \frac{1}{2}\int_V \frac{\partial u_i^e}{\partial x_j} D_{ijkl} \frac{\partial u_k}{\partial x_l} dV - \frac{1}{2}\int_V \frac{\partial u_i}{\partial x_j} D_{ijkl} \frac{\partial u_k^e}{\partial x_l} dV + \frac{1}{2}\int_V \frac{\partial u_i^e}{\partial x_j} D_{ijkl} \frac{\partial u_k^e}{\partial x_l} dV$$

$$- \int_V (u_i - u_i^e) b_i dV - \int_{S_t} (u_i - u_i^e)\overline{T}_i dS$$

$$= \frac{1}{2}\int_V \frac{\partial u_i}{\partial x_j} D_{ijkl} \frac{\partial u_k}{\partial x_l} dV - \int_V u_i b_i dV - \int_{S_t} u_i \overline{T}_i dS + \frac{1}{2}\int_V \frac{\partial u_i^e}{\partial x_j} D_{ijkl} \frac{\partial u_k^e}{\partial x_l} dV$$

$$- (\int_V \frac{\partial u_i^e}{\partial x_j} D_{ijkl} \frac{\partial u_k}{\partial x_l} dV - \int_V u_i^e b_i dV - \int_{S_t} u_i^e \overline{T}_i dS)$$

$$= \frac{1}{2}\int_V \varepsilon_{ij}\sigma_{ij} dV - \int_V u_i b_i dV - \int_{S_t} u_i \overline{T}_i dS + \frac{1}{2}\int_V \frac{\partial u_i^e}{\partial x_j} D_{ijkl} \frac{\partial u_k^e}{\partial x_l} dV -$$

$$(\int_V \frac{\partial u_i^e}{\partial x_j} \sigma_{ij} dV - \int_V u_i^e b_i dV - \int_{S_t} u_i^e \overline{T}_i dS)$$

$$= \Pi^E + \frac{1}{2}\int_V \frac{\partial u_i^e}{\partial x_j} D_{ijkl} \frac{\partial u_k^e}{\partial x_l} dV - (-\int_V u_i^e \sigma_{ij,j} dV + \int_S u_i^e \sigma_{ij} n_j dS - \int_V u_i^e b_i dV - \int_{S_t} u_i^e \overline{T}_i dS)$$

$$= \Pi^E + \frac{1}{2}\int_V \frac{\partial u_i^e}{\partial x_j} D_{ijkl} \frac{\partial u_k^e}{\partial x_l} dV - (-\int_V u_i^e \sigma_{ij,j} dV + \int_{S_t} u_i^e \overline{T}_i dS - \int_V u_i^e b_i dV - \int_{S_t} u_i^e \overline{T}_i dS)$$

$$= \Pi^E + \frac{1}{2}\int_V \frac{\partial u_i^e}{\partial x_j} D_{ijkl} \frac{\partial u_k^e}{\partial x_l} dV + (\int_V u_i^e \sigma_{ij,j} dV + \int_V u_i^e b_i dV)$$

$$= \Pi^E + \frac{1}{2}\int_V \frac{\partial u_i^e}{\partial x_j} D_{ijkl} \frac{\partial u_k^e}{\partial x_l} dV + \int_V u_i^e (\sigma_{ij,j} + b_i) dV$$

$$= \Pi^E + \frac{1}{2}\int_V \frac{\partial u_i^e}{\partial x_j} D_{ijkl} \frac{\partial u_k^e}{\partial x_l} dV$$

Since D_{ijkl} is positive definite, $\frac{\partial u_i^e}{\partial x_j} D_{ijkl} \frac{\partial u_k^e}{\partial x_l} > 0$ for all $\frac{\partial u_i^e}{\partial x_j} \neq 0$. Therefore, $\int_V \frac{\partial u_i^e}{\partial x_j} D_{ijkl} \frac{\partial u_k^e}{\partial x_l} dV > 0$ for all $\frac{\partial u_i^e}{\partial x_j} \neq 0$, and $\int_V \frac{\partial u_i^e}{\partial x_j} D_{ijkl} \frac{\partial u_k^e}{\partial x_l} dV = 0$ iff $\frac{\partial u_i^e}{\partial x_j} \equiv 0$.

$\Pi^h \geq \Pi^E$ (The equality sign holds only for $u_i^h = u_i + ??$.)

9.3. Principle of Virtual Work

- If the following inequality is valid for all real number α, the principle of virtual work holds.

$$\Pi^{RR}(u_i + \alpha v_i) \geq \Pi^{RR}(u_i) \quad \forall v_i \in \upsilon$$

$$g(\alpha) \equiv \Pi^{RR}(u_i + \alpha v_i)$$
$$= \frac{1}{2}\int_V \frac{\partial (u_i + \alpha v_i)}{\partial x_j} D_{ijkl} \frac{\partial (u_i + \alpha v_i)}{\partial x_l} dV - \int_V (u_i + \alpha v_i) b_i dV - \int_{\Gamma_t} (u_i + \alpha v_i)\overline{T}_i d\Gamma$$
$$= \frac{1}{2}\int_V \frac{\partial u_i}{\partial x_j} D_{ijkl} \frac{\partial u_k}{\partial x_l} dV + \alpha \int_V \frac{\partial v_i}{\partial x_j} D_{ijkl} \frac{\partial u_k}{\partial x_l} dV + \frac{1}{2}\alpha^2 \int_V \frac{\partial v_i}{\partial x_j} D_{ijkl} \frac{\partial v_i}{\partial x_l} dV$$
$$- \int_V (u_i + \alpha v_i) b_i dV - \int_{\Gamma_t} (u_i + \alpha v_i)\overline{T}_i d\Gamma$$

$$g'(\alpha) = \int_V \frac{\partial v_i}{\partial x_j} D_{ijkl} \frac{\partial u_k}{\partial x_l} dV - \int_V v_i b_i dV - \int_{\Gamma_t} v_i \overline{T}_i d\Gamma + \alpha \int_V \frac{\partial v_i}{\partial x_j} D_{ijkl} \frac{\partial v_i}{\partial x_l} dV$$

$$g'(0) = \int_V \frac{\partial v_i}{\partial x_j} D_{ijkl} \frac{\partial u_k}{\partial x_l} dV - \int_V v_i b_i dV - \int_{\Gamma_t} v_i \overline{T}_i d\Gamma = 0 \quad \forall v_i \in \upsilon \text{ (for all admissible } v_i \text{)}$$

$$\int_V \frac{\partial v_i}{\partial x_j} \sigma_{ij} dV - \int_V v_i b_i dV - \int_{\Gamma_t} v_i \overline{T}_i d\Gamma = 0 \quad \text{for all admissible } v_i$$

If the principle of virtual work holds, then the principle of minimum potential energy holds because the term in the parenthesis in the boxed equation of the principle of minimum potential energy vanishes identically. The approximate version of the principle of virtual work is

$$\int_V \frac{\partial v_i^h}{\partial x_j} \sigma_{ij}^h dV - \int_V v_i^h b_i dV - \int_{\Gamma_t} v_i^h \overline{T}_i d\Gamma = 0 \quad \text{for all admissible } v_i^h$$

9.4. Rayleigh-Ritz Type Discretization

- Approximation

$$u_i^h = c_{i1}g_1 + c_{i2}g_2 + \cdots + c_{in}g_n = \sum_{p=1}^{n} c_{ip}g_p$$

- Principle of Minimum Potential Energy

$$\frac{\partial u_i^h}{\partial x_j} = \sum_{p=1}^{n} c_{ip}\frac{\partial g_p}{\partial x_j} = c_{ip}\frac{\partial g_p}{\partial x_j}$$

$$\text{Min}\Pi^h = \frac{1}{2}\int_V c_{ip}\frac{\partial g_p}{\partial x_j} D_{ijkl} c_{kq}\frac{\partial g_q}{\partial x_l} dV - \int_V c_{ip} g_p b_i dV - \int_{\Gamma_t} c_{ip} g_p \overline{T}_i d\Gamma$$

or $\dfrac{\partial \Pi^h}{\partial c_{mr}} = 0$ for all m, r

$$\frac{\partial \Pi^h}{\partial c_{mr}} = \frac{1}{2}\int_V \delta_{mi}\frac{\partial g_r}{\partial x_j} D_{ijkl} c_{kq}\frac{\partial g_q}{\partial x_l} dV + \frac{1}{2}\int_V c_{ip}\frac{\partial g_p}{\partial x_j} D_{ijkl}\delta_{mk}\frac{\partial g_r}{\partial x_l} dV - \int_V \delta_{mi} g_r b_i dV - \int_{\Gamma_t} \delta_{mi} g_r \overline{T}_i d\Gamma$$

$$= \frac{1}{2}\int_V \frac{\partial g_r}{\partial x_j} D_{mjkl} c_{kq}\frac{\partial g_q}{\partial x_l} dV + \frac{1}{2}\int_V c_{ip}\frac{\partial g_p}{\partial x_j} D_{ijml}\frac{\partial g_r}{\partial x_l} dV - \int_V g_r b_m dV - \int_{\Gamma_t} g_r \overline{T}_m d\Gamma$$

$$= \int_V \frac{\partial g_r}{\partial x_j} D_{mjkl} c_{kp}\frac{\partial g_p}{\partial x_l} dV - \int_V g_r b_m dV - \int_{\Gamma_t} g_r \overline{T}_m d\Gamma$$

$$= \int_V \frac{\partial g_r}{\partial x_j} D_{mjkl}\frac{\partial g_p}{\partial x_l} dV c_{kp} - \int_V g_r b_m dV - \int_{\Gamma_t} g_r \overline{T}_m d\Gamma$$

$$= K_{rmkp}c_{kp} - f_{rm} = 0 \quad \text{for all } r \text{ and } m$$

- Principle of Virtual Work

$$v_i^h = \delta u_i^h = \delta c_{i1}g_1 + \delta c_{i2}g_2 + \cdots + \delta c_{in}g_n = \sum_{p=1}^{n}\delta c_{ip}g_p = \delta c_{ip}g_p$$

$$\delta\Pi^h = \int_V \delta c_{ip}\frac{\partial g_p}{\partial x_j} D_{ijkl} c_{kq}\frac{\partial g_q}{\partial x_l} dV - \int_V \delta c_{ip} g_p b_i dV - \int_{\Gamma_t} \delta c_{ip} g_p \overline{T}_i d\Gamma$$

$$= \delta c_{ip}\left(\int_V \frac{\partial g_p}{\partial x_j} D_{ijkl}\frac{\partial g_q}{\partial x_l} dV c_{kq} - \int_V g_p b_i dV - \int_{\Gamma_t} g_p \overline{T}_i d\Gamma\right)$$

$$= \delta c_{ip}\left(\int_V \frac{\partial g_p}{\partial x_j} D_{ijkl}\frac{\partial g_q}{\partial x_l} dV c_{kq} - \int_V g_p b_i dV - \int_{\Gamma_t} g_p \overline{T}_i d\Gamma\right) = 0 \quad \text{for all } \delta c_{ip}$$

$$\delta\Pi^h = K_{pikq}c_{kq} - f_{pi} = 0 \quad \text{for all } p \text{ and } i$$

- Matrix Form – Virtual Work Expression

$$\int_V \delta\varepsilon_{ij}^h \sigma_{ij}^h dV - \int_V \delta u_i^h b_i dV - \int_{\Gamma_t} \delta u_i^h \overline{T}_i d\Gamma = 0$$

$$\int_V \delta\varepsilon_{ij}^h \sigma_{ij}^h dV =$$

$$\int_V (\delta\varepsilon_{11}^h \sigma_{11}^h + \delta\varepsilon_{22}^h \sigma_{22}^h + \delta\varepsilon_{33}^h \sigma_{33}^h + \delta\varepsilon_{12}^h \sigma_{12}^h + \delta\varepsilon_{21}^h \sigma_{21}^h + \delta\varepsilon_{13}^h \sigma_{13}^h + \delta\varepsilon_{31}^h \sigma_{31}^h + \delta\varepsilon_{23}^h \sigma_{23}^h + \delta\varepsilon_{32}^h \sigma_{32}^h) dV =$$

$$\int_V (\delta\varepsilon_{11}^h \sigma_{11}^h + \delta\varepsilon_{22}^h \sigma_{22}^h + \delta\varepsilon_{33}^h \sigma_{33}^h + 2\delta\varepsilon_{12}^h \sigma_{12}^h + 2\delta\varepsilon_{13}^h \sigma_{13}^h + 2\delta\varepsilon_{23}^h \sigma_{23}^h) dV =$$

$$\int_V (\delta\varepsilon_{11}^h \sigma_{11}^h + \delta\varepsilon_{22}^h \sigma_{22}^h + \delta\varepsilon_{33}^h \sigma_{33}^h + \delta\gamma_{12}^h \sigma_{12}^h + \delta\gamma_{13}^h \sigma_{13}^h + \delta\gamma_{23}^h \sigma_{23}^h) dV = \int_V \delta\boldsymbol{\varepsilon}^{h^T} \boldsymbol{\sigma}^h dV$$

$$\underline{\underline{\int_V \delta\boldsymbol{\varepsilon}^h \cdot \boldsymbol{\sigma}^h dV = \int_V \delta\mathbf{u}^h \cdot \mathbf{b}\ dV + \int_{\Gamma_t} \delta\mathbf{u}^h \cdot \overline{\mathbf{T}} d\Gamma}}$$

- Displacement

$$\mathbf{u}^h = \begin{pmatrix} u_1^h \\ u_2^h \\ u_3^h \end{pmatrix} = \begin{bmatrix} g_1 & 0 & 0 & g_2 & 0 & 0 & & g_n & 0 & 0 \\ 0 & g_1 & 0 & 0 & g_2 & 0 & \cdots & 0 & g_n & 0 \\ 0 & 0 & g_1 & 0 & 0 & g_2 & & 0 & 0 & g_n \end{bmatrix} \begin{pmatrix} c_{11} \\ c_{21} \\ c_{31} \\ c_{12} \\ c_{22} \\ c_{32} \\ \vdots \\ c_{1n} \\ c_{2n} \\ c_{3n} \end{pmatrix} = \mathbf{Nc}$$

- Virtual Strain

$$(\delta\varepsilon^h) = \begin{pmatrix} \delta\varepsilon_{11}^h \\ \delta\varepsilon_{22}^h \\ \delta\varepsilon_{33}^h \\ \delta\gamma_{12}^h \\ \delta\gamma_{13}^h \\ \delta\gamma_{23}^h \end{pmatrix} = \begin{pmatrix} \dfrac{\partial \delta u_1^h}{\partial x_1} \\ \dfrac{\partial \delta u_2^h}{\partial x_2} \\ \dfrac{\partial \delta u_3^h}{\partial x_3} \\ \dfrac{\partial \delta u_1^h}{\partial x_2} + \dfrac{\partial \delta u_2^h}{\partial x_1} \\ \dfrac{\partial \delta u_1^h}{\partial x_3} + \dfrac{\partial \delta u_3^h}{\partial x_1} \\ \dfrac{\partial \delta u_2^h}{\partial x_3} + \dfrac{\partial \delta u_3^h}{\partial x_2} \end{pmatrix} = \begin{pmatrix} \delta c_{1i} \dfrac{\partial g_i}{\partial x_1} \\ \delta c_{2i} \dfrac{\partial g_i}{\partial x_2} \\ \delta c_{3i} \dfrac{\partial g_i}{\partial x_3} \\ \delta c_{1i} \dfrac{\partial g_i}{\partial x_2} + \delta c_{2i} \dfrac{\partial g_i}{\partial x_1} \\ \delta c_{1i} \dfrac{\partial g_i}{\partial x_3} + \delta c_{3i} \dfrac{\partial g_i}{\partial x_1} \\ \delta c_{2i} \dfrac{\partial g_i}{\partial x_3} + \delta c_{3i} \dfrac{\partial g_i}{\partial x_2} \end{pmatrix}$$

$$= \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & 0 & 0 & \frac{\partial g_2}{\partial x_1} & 0 & 0 & & \frac{\partial g_n}{\partial x_1} & 0 & 0 \\ 0 & \frac{\partial g_1}{\partial x_2} & 0 & 0 & \frac{\partial g_2}{\partial x_2} & 0 & & 0 & \frac{\partial g_n}{\partial x_2} & 0 \\ 0 & 0 & \frac{\partial g_1}{\partial x_3} & 0 & 0 & \frac{\partial g_2}{\partial x_3} & \cdots & 0 & 0 & \frac{\partial g_n}{\partial x_3} \\ \frac{\partial g_1}{\partial x_2} & \frac{\partial g_1}{\partial x_1} & 0 & \frac{\partial g_2}{\partial x_2} & \frac{\partial g_2}{\partial x_1} & 0 & & \frac{\partial g_n}{\partial x_2} & \frac{\partial g_n}{\partial x_1} & 0 \\ \frac{\partial g_1}{\partial x_3} & 0 & \frac{\partial g_1}{\partial x_1} & \frac{\partial g_2}{\partial x_3} & 0 & \frac{\partial g_2}{\partial x_1} & & \frac{\partial g_n}{\partial x_3} & 0 & \frac{\partial g_n}{\partial x_1} \\ 0 & \frac{\partial g_1}{\partial x_3} & \frac{\partial g_1}{\partial x_2} & 0 & \frac{\partial g_2}{\partial x_3} & \frac{\partial g_2}{\partial x_2} & & 0 & \frac{\partial g_n}{\partial x_3} & \frac{\partial g_n}{\partial x_2} \end{bmatrix} \begin{pmatrix} \delta c_{11} \\ \delta c_{21} \\ \delta c_{31} \\ \delta c_{12} \\ \delta c_{22} \\ \delta c_{32} \\ \vdots \\ \delta c_{1n} \\ \delta c_{2n} \\ \delta c_{3n} \end{pmatrix} = \mathbf{B}\delta\mathbf{c}$$

- Stress-strain (displacement) Relation

$$(\sigma^h) = \begin{pmatrix} \sigma_{11}^h \\ \sigma_{22}^h \\ \sigma_{33}^h \\ \sigma_{12}^h \\ \sigma_{13}^h \\ \sigma_{13}^h \end{pmatrix} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 0 & 0 & 0 \\ \frac{\nu}{1-\nu} & 1 & \frac{\nu}{1-\nu} & 0 & 0 & 0 \\ \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} \end{bmatrix} \begin{pmatrix} \varepsilon_{11}^h \\ \varepsilon_{22}^h \\ \varepsilon_{33}^h \\ \gamma_{12}^h \\ \gamma_{13}^h \\ \gamma_{23}^h \end{pmatrix}$$

$$= \mathbf{D}\boldsymbol{\varepsilon}^h = \mathbf{DBc}$$

- Final System Equation

$$\int_V \varepsilon_{ij}^h \sigma_{ij}^h dV = \int_V \boldsymbol{\varepsilon}^{h^T} \cdot \boldsymbol{\sigma}^h dV = \mathbf{c}^T \int_V \mathbf{B}^T \mathbf{DB} dV \mathbf{c} = \mathbf{c}^T \mathbf{Kc}$$

$$\int_V u_i^h b_i dV = \int_V \mathbf{u}^{h^T} \mathbf{b} dV = \mathbf{c}^T \int_V \mathbf{N}^T \mathbf{b} dV = \mathbf{c}^T \cdot \mathbf{f}_b$$

$$\int_{\Gamma_t} u_i^h \overline{T}_i d\Gamma = \int_{\Gamma_t} \mathbf{u}^{h^T} \overline{\mathbf{T}} d\Gamma = \mathbf{c}^T \int_{\Gamma_t} \mathbf{N}^T \overline{\mathbf{T}} d\Gamma = \mathbf{c}^T \cdot \mathbf{f}_t$$

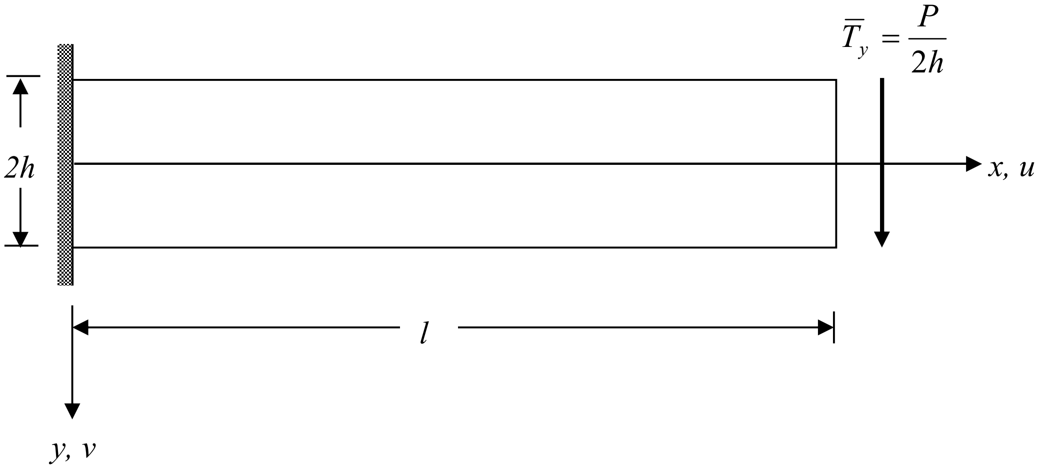
$$\rightarrow \quad \Pi^h = \frac{1}{2} \int_V \boldsymbol{\varepsilon}^h \cdot \boldsymbol{\sigma}^h dV - \int_V \mathbf{u}^h \cdot \mathbf{b} \, dV - \int_{\Gamma_t} \mathbf{u}^h \cdot \overline{\mathbf{T}} d\Gamma$$

$$= \frac{1}{2} \mathbf{c}^T \mathbf{Kc} - \mathbf{c}^T \cdot \mathbf{f}_b - \mathbf{c}^T \cdot \mathbf{f}_t = \frac{1}{2} \mathbf{c}^T \mathbf{Kc} - \mathbf{c}^T \cdot \mathbf{f}$$

- Principle of Minimun Potential Energy

$$\nabla \Pi^h = \frac{\partial \Pi^h}{\partial \mathbf{c}} = \mathbf{Kc} - \mathbf{f} = 0 \rightarrow \mathbf{Kc} = \mathbf{f}$$

9.5 Example



By the elementary beam solution, the displacement field of the structure is assumed as

$$u = a\left(\frac{x^2}{2} - lx\right)y$$

$$v = b\left(\frac{x^3}{6} - \frac{x^2}{2}l\right)$$

$$\mathbf{N} = \begin{bmatrix} \left(\frac{x^2}{2} - lx\right)y & 0 \\ 0 & \frac{x^6}{6} - \frac{x^2 l}{2} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} (x-l)y & 0 \\ 0 & 0 \\ \frac{x^2}{2} - lx & \frac{x^2}{2} - lx \end{bmatrix}, \quad \mathbf{D} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

$$\mathbf{K} = \frac{E}{1-\nu^2}\int_0^l \int_{-h}^h \int_0^1 \begin{bmatrix} (x-l)y & 0 & \frac{x^2}{2} - lx \\ 0 & 0 & \frac{x^2}{2} - lx \end{bmatrix} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} (x-l)y & 0 \\ 0 & 0 \\ \frac{x^2}{2} - lx & \frac{x^2}{2} - lx \end{bmatrix} dzdydx$$

$$= \frac{E}{1-\nu^2}\int_0^l \int_{-h}^h \int_0^1 \begin{bmatrix} (x-l)^2 y^2 + \frac{1-\nu}{2}\left(\frac{x^2}{2} - lx\right)^2 & \frac{1-\nu}{2}\left(\frac{x^2}{2} - lx\right)^2 \\ \frac{1-\nu}{2}\left(\frac{x^2}{2} - lx\right)^2 & \frac{1-\nu}{2}\left(\frac{x^2}{2} - lx\right)^2 \end{bmatrix} dzdydx$$

$$= \frac{E}{1-\nu^2}\begin{bmatrix} 2\frac{l^3}{3}\frac{h^3}{3} + \frac{1-\nu}{2}\frac{2}{15}l^5(2h) & \frac{1-\nu}{2}\frac{2}{15}l^5(2h) \\ \frac{1-\nu}{2}\frac{2}{15}l^5(2h) & \frac{1-\nu}{2}\frac{2}{15}l^5(2h) \end{bmatrix}$$

$$\mathbf{f} = \int_{-h}^h \int_0^1 \begin{bmatrix} -\frac{l^2}{2}y & 0 \\ 0 & -\frac{l^3}{3} \end{bmatrix} \begin{pmatrix} 0 \\ \frac{P}{2h} \end{pmatrix} dy = -\frac{l^3}{3}\begin{pmatrix} 0 \\ P \end{pmatrix}$$

$$\frac{E}{1-\nu^2}\begin{bmatrix} 2\frac{l^3}{3}\frac{h^3}{3}+\frac{1-\nu}{2}\frac{2}{15}l^5(2h) & \frac{1-\nu}{2}\frac{2}{15}l^5(2h) \\ \frac{1-\nu}{2}\frac{2}{15}l^5(2h) & \frac{1-\nu}{2}\frac{2}{15}l^5(2h) \end{bmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -\frac{l^3}{3}\begin{pmatrix} 0 \\ P \end{pmatrix}$$

$$a = \frac{3}{2}\frac{1-\nu^2}{Eh^3}P = \frac{1-\nu^2}{EI}P$$

$$b = -(1+\frac{5}{3}\frac{1}{1-\nu}(\frac{h}{l})^2)\frac{1-\nu^2}{EI}P$$

$$u = \frac{1-\nu^2}{EI}P(\frac{x^2}{2}-lx)y$$

$$v = -(1+\frac{5}{3}\frac{1}{1-\nu}(\frac{h}{l})^2)\frac{1-\nu^2}{EI}P(\frac{x^3}{6}-\frac{x^2}{2}l)$$

$$\sigma_{xx} = \frac{P(x-l)}{I}y = \frac{M}{I}y$$

$$\sigma_{yy} = \nu\frac{M}{I}y$$

$$\tau_{xy} = -\frac{5}{6}(\frac{h}{l})^2\frac{1}{I}(\frac{x^2}{2}-xl)P$$