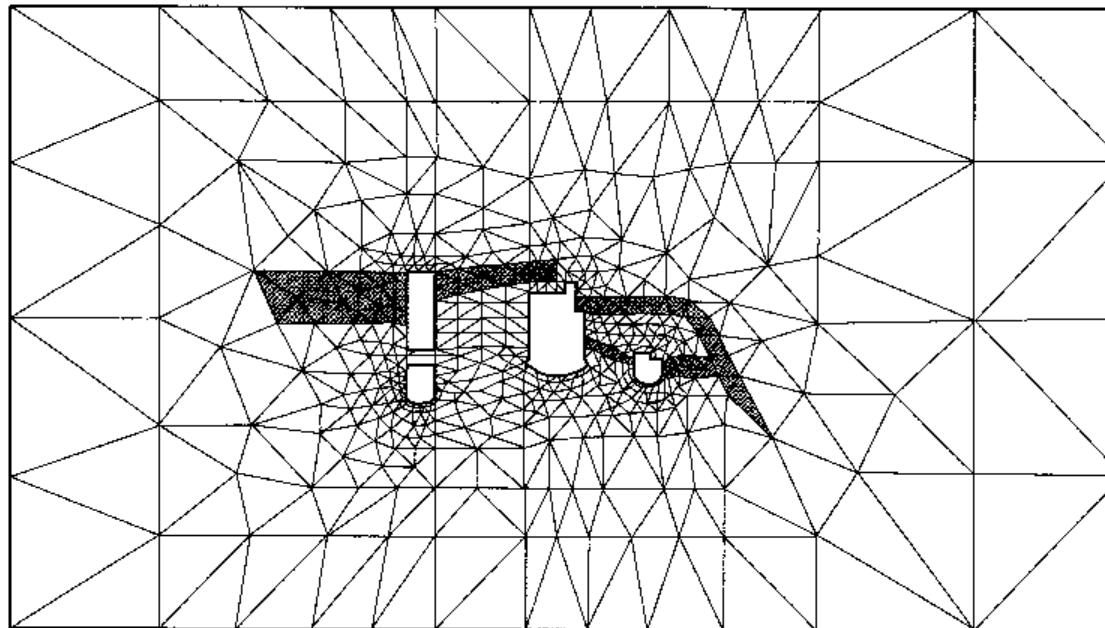


이 문서는 서울대학교 공과대학 건설환경공학과 이해성 교수가 1993 ~ 2022년 까지 약 30년 간에 걸쳐 강의한 유한요소법입문 강좌를 위하여 제작한 강의용 슬라이드입니다. 이 슬라이드는 비상업적 교육용으로 누구나 사용하고 배포할 수 있으나, 상업적 목적으로는 사용하기 위하여는 저작자의 사전 동의를 받아야 합니다. 또한 이 자료의 내용을 인용할 경우에는 적절한 방법을 통하여 출처를 반드시 명기하여야 합니다.

Introduction to Finite Element Method



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Class Contents

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Chapter 2

Approximation of Functions and Variational Calculus

$$\delta I = \int_0^l \left(\frac{\partial F}{\partial f} - \frac{d}{dx} \frac{\partial F}{\partial f'} \right) g_k dx$$

\approx

$$\delta I = \int_0^l \left(\frac{\partial F}{\partial f} g_k + \frac{\partial F}{\partial f'} g'_k \right) dx$$
$$= \frac{\partial F}{\partial f} g_k + \int_0^l \left(\frac{\partial F}{\partial f'} g'_k \right) dx$$

In case the variation vanishes at the boundaries, then

$$= \int_0^l \left(\frac{\partial F}{\partial f} - \frac{d}{dx} \frac{\partial F}{\partial f'} \right) g_k dx$$

functions vanish at the boundary

$$\frac{d}{dx} \frac{\partial F}{\partial f'} = 0 \quad \text{for all } k \Leftrightarrow \frac{\partial F}{\partial f'} = 0$$

Fundamental Considerations

- **What is the best solution for a given problem ?**

Needless to say, it is the exact solution...

- **What is the exact solution?**

The solution that satisfies the governing equations as well as boundary conditions if any.

- **How many do the exact solutions exist?**

Of course, one... In case several or infinite numbers of the exact solutions exist for a given equation, we call the problem as an ill-posed problem, and have great difficulties in determining a solution of the given problem...

- **What if it is impossible to determine the exact solution for various reasons?**

We need approximate solutions.

- **What is an approximate solution?**

- **Fundamental Questions**

- What is the best approximation?
- How can we represent the best approximation?

- **A Good (or Robust or Well Formulated) Approximation Should**

- Yield the best approximation to the exact solution for a given degree of approximation.
- Converge to the exact solution as higher degree of approximation is employed.

- **What is the Definition of the Best Approximation?**

- May be defined as the closest solution to the exact solution.
- But, how close is “the closest”?
- “Close or Far” implies the distance between two spatial points.
- We should define some sort of ruler to measure the distance between two elements in a function set...

- **Norms of Functions:** A measure of a function set

A function set \mathcal{V} is said to be a *normed space* if to every $f \in \mathcal{V}$ there is associated a nonnegative real number $\|f\|$, called the norm of f , in such way that

- $\|f\| = 0$ if and only if $f \equiv 0$
- $\|\alpha f\| = |\alpha| \|f\|$ for any real number α .
- $\|f + g\| \leq \|f\| + \|g\|$

Every normed space may be regarded as a metric space, in which the distance between any two elements in the space is measured by the defined norm. Various types of norm can be defined for a function space. Among them the following norms are important.

- L_1 norm: $\|f\|_{L_1} = \int_V |f| dV$
- L_2 norm: $\|f\|_{L_2} = \left(\int_V f^2 dV \right)^{1/2}$
- H^1 norm: $\|f\|_{H^1} = \left(\int_V (f^2 + \nabla f \cdot \nabla f) dV \right)^{1/2}$

- **Discretization**

- Representation of a continuously distributed quantities with some numbers.

$$f(\mathbf{X}) = \sum_{i=1}^n a_i g_i(\mathbf{X}) \quad \forall f(\mathbf{X}) \in \mathcal{V} \quad \text{where } g_i \text{ are the basis functions of a function set, } \mathcal{V}.$$

- Set : Collection of some objectives with the same characteristics
- The basis functions should be linearly independent to each other.

$$\sum_{i=1}^n a_i g_i(\mathbf{X}) = 0 \quad \text{if and only if all } a_i = 0$$

- Taylor series, Fourier series, etc...

- **Approximations - Truncation**

$$f(\mathbf{X}) = \sum_{i=1}^n a_i g_i(\mathbf{X}) \approx f^h(\mathbf{X}) = \sum_{i=1}^m a_i g_i(\mathbf{X}) \in \mathcal{V}^h \subset \mathcal{V} \quad \text{where } m \leq n$$

- **Summation Notation:** Repeated indices denote summation $\sum_{i=1}^m a_i b_i = a_i b_i$.

- **General Ideas for the Best Approximation**

Let's find out a approximate function that is closest to the given function by use of a norm defined in the function space. If this is the case, the characteristics of an approximation method depend on those of the norm used in the approximation.

- **Least Square Error Minimization**

Error: $e = f - f^h$

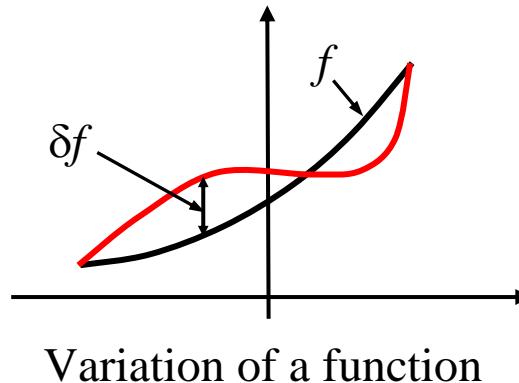
$$\text{Minimize } \Pi = \frac{1}{2} \|e\|_{L_2}^2 = \frac{1}{2} \|f - f^h\|_{L_2}^2 = \frac{1}{2} \int_V (f - f^h)^2 dV$$

$$\begin{aligned} \frac{\partial \Pi}{\partial a_k} &= \int_V (f^h - f) \frac{\partial f^h}{\partial a_k} dV = \int_V (f^h - f) g_k dV = \int_V g_k \sum_{i=1}^m g_i a_i dV - \int_V g_k f dV \\ &= \sum_{i=1}^m \int_V g_k g_i dV a_i - \int_V g_k f dV = \sum_{i=1}^m K_{ki} a_i - F_k = 0 \text{ for } k = 1, \dots, m \text{ or } \mathbf{Ka} = \mathbf{F} \end{aligned}$$

If the basis functions are orthogonal, \mathbf{K} becomes diagonal.

- **Variation of a function**

- *The variation of a function means a possible change in the function for the fixed x.*



- **Variational Calculus**

- if $f = a_i g_i$ $\delta f = \delta a_i g_i$ or $\delta f = \frac{\partial f}{\partial a_i} \delta a_i$.
- $F(f) : \delta F = \frac{\partial F}{\partial a_i} \delta a_i = \frac{\partial F}{\partial f} \frac{\partial f}{\partial a_i} \delta a_i = \frac{\partial F}{\partial f} \delta f$
- $\delta(f + h) = \delta f + \delta h$, $\delta(fh) = h\delta f + f\delta h$
- $\delta \frac{df}{dx} = \frac{\partial}{\partial a_i} \left(\frac{df}{dx} \right) \delta a_i = \frac{d}{dx} \left(\frac{\partial f}{\partial a_i} \delta a_i \right) = \frac{d\delta f}{dx}$, $\delta \int f dx = \frac{\partial}{\partial a_i} \int f dx \delta a_i = \int \frac{\partial f}{\partial a_i} \delta a_i dx = \int \delta f dx$

- **Minimization by Variational Calculus**

$$\text{Min } \Pi(f^h) = \frac{1}{2} \int_0^l (f - f^h)^2 dx$$

$$\delta\Pi(f^h) = \delta\left(\frac{1}{2} \int_0^l (f^h - f)^2 dx\right) = \frac{1}{2} \int_0^l \delta(f^h - f)^2 dx = \int_0^l (f^h - f) \delta f^h dx$$

$$= \int_0^l (f^h - f) \frac{\partial f^h}{\partial a_k} dx \delta a_k = \frac{\partial \Pi}{\partial a_k} \delta a_k$$

$$\text{Min } \Pi(f^h) = \frac{1}{2} \int_0^l (f - f^h)^2 dx \Leftrightarrow \delta\Pi = 0 \text{ for all possible } \delta a_k$$

- **Euler Equation**

$$\text{Min } \Pi(f) = \int_0^l F(f, f', x) dx \Rightarrow \frac{\partial \Pi}{\partial a_k} = 0 \text{ for all } k$$

$$\frac{\partial \Pi}{\partial a_k} = \frac{\partial}{\partial a_k} \int_0^l F(f, f', x) dx = \int_0^l \frac{\partial F}{\partial a_k} dx = \int_0^l \left(\frac{\partial F}{\partial f} \frac{\partial f}{\partial a_k} + \frac{\partial F}{\partial f'} \frac{\partial f'}{\partial a_k} \right) dx = \int_0^l \left(\frac{\partial F}{\partial f} g_k + \frac{\partial F}{\partial f'} g'_k \right) dx$$

$$= \frac{\partial F}{\partial f'} g_k \Big|_0^l + \int_0^l \left(\frac{\partial F}{\partial f} g_k - \frac{d}{dx} \frac{\partial F}{\partial f'} g_k \right) dx$$

In case the **basis functions vanish** at the boundary, then

$$\frac{\partial \Pi}{\partial a_k} = \int_0^l \left(\frac{\partial F}{\partial f} - \frac{d}{dx} \frac{\partial F}{\partial f'} \right) g_k dx = 0 \quad \text{for all } k \Leftrightarrow \frac{\partial F}{\partial f} - \frac{d}{dx} \frac{\partial F}{\partial f'} = 0$$

$$\delta \Pi(f) = \int_0^l \delta F(f, f', x) dx = \int_0^l \left(\frac{\partial F}{\partial f} \delta f + \frac{\partial F}{\partial f'} \delta f' \right) dx = \frac{\partial F}{\partial f'} \delta f \Big|_0^l + \int_0^l \left(\frac{\partial F}{\partial f} \delta f - \frac{d}{dx} \frac{\partial F}{\partial f'} \delta f' \right) dx$$

In case the variation vanishes at the boundaries, then

$$\delta \Pi = \int_0^l \left(\frac{\partial F}{\partial f} - \frac{d}{dx} \frac{\partial F}{\partial f'} \right) \delta f dx = \int_0^l \left(\frac{\partial F}{\partial f} - \frac{d}{dx} \frac{\partial F}{\partial f'} \right) g_k dx \delta a_k = \frac{\partial \Pi}{\partial a_k} \delta a_k$$

Therefore,

$$\text{Min } \Pi \Leftrightarrow \delta \Pi = 0$$

- **Example 1**

$$\text{Min } \Pi(y) = \int_{x_1}^{x_2} \sqrt{1 + (y')^2} dx \text{ subject to } y(x_1) = y_1, \quad y(x_2) = y_2$$

$$\frac{\partial F}{\partial f} - \frac{d}{dx} \frac{\partial F}{\partial f'} = -\frac{d}{dx} y' (1 + (y')^2)^{-1/2} = 0$$

$$\frac{d}{dx} y' (1 + (y')^2)^{-1/2} = y'' (1 + (y')^2)^{-1/2} + y' \left(-\frac{1}{2}\right) (1 + (y')^2)^{-3/2} y' y''$$

$$= y'' (1 + (y')^2)^{-1/2} \left(1 - \frac{(y')^2}{1 + (y')^2}\right) = y'' (1 + (y')^2)^{-3/2} = 0$$

$$y'' = 0 \rightarrow y = ax + b. \quad \text{By applying BC, } y'' = 0 \rightarrow y = \frac{y_2 - y_1}{x_2 - x_1} x + \frac{x_2 y_1 - x_1 y_2}{x_2 - x_1}$$

- **Example 2**

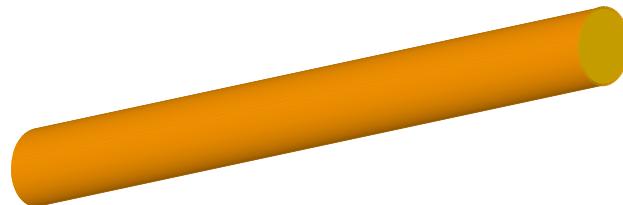
$$\text{Min } \Pi(u) = \int_0^l \left(\frac{1}{2}(u')^2 - uf\right) dx \text{ subject to } u(0) = 0, \quad u(l) = 0$$

$$\frac{\partial F}{\partial u} - \frac{d}{dx} \frac{\partial F}{\partial u'} = -f - \frac{d}{dx} \left(\frac{\partial}{\partial u'} \frac{1}{2}(u')^2\right) = -f - \frac{d}{dx} u' = -f - u'' = 0 \rightarrow u'' + f = 0$$

Homework 1

Chapter 3

Elliptic Differential Equations In One Dimension



$$\begin{aligned}
 & \int_{x_0}^{x_1} \frac{d\sigma}{dx} \frac{\partial f}{\partial x} dx U_j = \\
 & \quad \delta U_1 \int_{x_1}^{x_2} \frac{dg_1}{dx} \left(\frac{dg_1}{dx} U_1 + \right. \\
 & \quad \left. \left(\frac{dg_1}{dx} U_1 + \frac{dg_2}{dx} U_2 \right) dx \right) + \delta U_2 \int_{x_2}^{x_3} \frac{dg_2}{dx} \left(\frac{dg_2}{dx} U_2 + \frac{dg_3}{dx} U_3 \right. \\
 & \quad \left. \left. + \dots \right. \right. \\
 & \quad \left. \left. \frac{dg_{j-1}}{dx} \left(\frac{dg_{j-1}}{dx} U_{j-1} + \frac{dg_{j+1}}{dx} U_{j+1} \right) dx + \delta U_{j-1} \int_{x_{j-1}}^{x_j} \frac{dg_{j-1}}{dx} \left(\frac{dg_{j-1}}{dx} U_{j-1} + \right. \right. \\
 & \quad \left. \left. \frac{dg_j}{dx} U_j + \frac{dg_{j+1}}{dx} U_{j+1} \right) dx \right) + \delta U_j \int_{x_j}^{x_{j+1}} \frac{dg_j}{dx} \left(\frac{dg_j}{dx} U_j + \frac{dg_{j+1}}{dx} U_{j+1} \right. \\
 & \quad \left. \left. + \dots \right. \right. \\
 & \quad \left. \left. \frac{dg_{j+1}}{dx} U_{j+1} \right) dx \right) + \delta U_{j+1} \int_{x_{j+1}}^{x_{j+2}} \frac{dg_{j+1}}{dx} \left(\frac{dg_{j+1}}{dx} U_{j+1} + d\sigma \right)
 \end{aligned}$$

3.1 Problems with Homogenous Displacement BC

- **Problem Definition**

$$\frac{d^2u}{dx^2} + f = 0 \quad 0 < x < l, \quad u(0) = u(l) = 0$$

- **Approximation – Discretization**

$$u^h = \sum_{i=1}^m a_i g_i \quad \text{where} \quad u^h(0) = u^h(l) = 0$$

- **Residuals**

Verbal Definition : Something left over, or resulting from subtraction...

Equation Residual : $R_E = \frac{d^2u^h}{dx^2} + f \neq 0 \quad 0 < x < l$

Function Residual : $R_F = u - u^h \neq 0 \quad 0 < x < l$

- **Error Estimator :**

$$\Pi^R = \int_0^l R_F R_E dx = \int_0^l (u - u^h) \left(\frac{d^2u^h}{dx^2} + f \right) dx$$

- Least Square Error

$$\begin{aligned}\Pi^R &= \frac{1}{2} \int_0^l (u - u^h) \left(\frac{d^2 u^h}{dx^2} + f \right) dx = \frac{1}{2} \int_0^l (u - u^h) \left(\frac{d^2 u^h}{dx^2} - \frac{d^2 u}{dx^2} \right) dx \\ &= \frac{1}{2} (u - u^h) \left(\frac{du^h}{dx} - \frac{du}{dx} \right) \Big|_0^l - \frac{1}{2} \int_0^l \left(\frac{du}{dx} - \frac{du^h}{dx} \right) \left(\frac{du^h}{dx} - \frac{du}{dx} \right) dx = \boxed{\frac{1}{2} \int_0^l \left(\frac{du}{dx} - \frac{du^h}{dx} \right) \left(\frac{du}{dx} - \frac{du^h}{dx} \right) dx}\end{aligned}$$

- Energy Functional – Total potential Energy

$$\begin{aligned}\Pi^R &= \frac{1}{2} \int_0^l (u - u^h) \left(\frac{d^2 u^h}{dx^2} + f \right) dx = \frac{1}{2} \int_0^l \left(u \frac{d^2 u^h}{dx^2} + uf - u^h \frac{d^2 u^h}{dx^2} - u^h f \right) dx \\ &= \frac{1}{2} \left\{ \int_0^l (uf - u^h f) dx + u \frac{du^h}{dx} \Big|_0^l - \frac{du}{dx} u^h \Big|_0^l + \int_0^l \frac{d^2 u}{dx^2} u^h dx - u^h \frac{du^h}{dx} \Big|_0^l + \int_0^l \frac{du^h}{dx} \frac{du^h}{dx} dx \right\} \\ &= \frac{1}{2} \int_0^l u f dx + \left(\frac{1}{2} \int_0^l \frac{du^h}{dx} \frac{du^h}{dx} dx - \int_0^l u^h f dx \right) = C + \Pi^{RR}\end{aligned}$$

-f

- Minimization Problems

$$\text{Min } \Pi^R \Leftrightarrow \text{Min } \Pi^{LS} \Leftrightarrow \text{Min } \Pi^{RR} \text{ w.r.t. } u^h \in \mathcal{V}^h$$

- Min Π^{RR} : **Rayleigh-Ritz Method** or **Principle of Minimum Potential Energy**

- 1st Order Necessary Condition of Minimization Problem

$$\begin{aligned}\frac{\partial \Pi^{RR}}{\partial a_k} &= \int_0^l \frac{d}{da_k} \left(\frac{du^h}{dx} \right) \frac{du^h}{dx} dx - \int_0^l \frac{du^h}{da_k} f dx \\ &= \int_0^l \frac{d}{da_k} \left(\sum_{i=1}^m a_i \frac{dg_i}{dx} \right) \left(\sum_{i=1}^m a_i \frac{dg_i}{dx} \right) dx - \int_0^l \frac{d}{da_k} \left(\sum_{i=1}^m a_i g_i \right) f dx \\ &= \sum_{i=1}^m \int_0^l \frac{dg_k}{dx} \frac{dg_i}{dx} dx a_i - \int_0^l g_k f dx = \sum_{i=1}^m K_{ki} a_i - F_k = 0 \quad \text{for } k = 1, \dots, m \rightarrow \mathbf{Ka} = \mathbf{F}\end{aligned}$$

- $\delta \Pi^{RR} = 0$: **Variational Principle** or **Principle of Virtual Work**

$$\begin{aligned}\delta \Pi^{RR} &= \delta \left(\frac{1}{2} \int_0^l \frac{du^h}{dx} \frac{du^h}{dx} dx - \int_0^l u^h f dx \right) = \int_0^l \frac{du^h}{dx} \delta \frac{du^h}{dx} dx - \int_0^l \delta u^h f dx = \int_0^l \frac{d\delta u^h}{dx} \frac{du^h}{dx} dx - \int_0^l \delta u^h f dx \\ &= \sum_{k=1}^m \delta a_k \left(\sum_{i=1}^m \int_0^l \frac{dg_k}{dx} \frac{dg_i}{dx} dx a_i - \int_0^l g_k f dx \right) \\ &= \sum_{k=1}^m \delta a_k \left(\sum_{i=1}^m K_{ki} a_i - F_k \right) = 0 \rightarrow (\delta \mathbf{a})^T (\mathbf{Ka} - \mathbf{F}) = 0\end{aligned}$$

- **Solution Space**

- $u \in \mathcal{V} \equiv \{u \mid u(0) = u(l) = 0, \int_0^l (\frac{du}{dx})^2 dx < \infty\}$
- $\mathcal{V}^h \equiv \mathcal{V}$: The exact solution
- $\mathcal{V}^h \subset \mathcal{V}$: An approximate solution

- **Properties of K**

- Symmetry : $K_{ij} = \int_0^l \frac{dg_i}{dx} \frac{dg_j}{dx} dx = \int_0^l \frac{dg_j}{dx} \frac{dg_i}{dx} dx = K_{ji}$
- Positive Definiteness :

$$\begin{aligned} \int_0^l (\frac{du^h}{dx})^2 dx &= \int_0^l \sum_{i=1}^m \frac{dg_i}{dx} a_i \sum_{j=1}^m \frac{dg_j}{dx} a_j dx = \\ \sum_{i=1}^m a_i \sum_{j=1}^m \int_0^l \frac{dg_i}{dx} \frac{dg_j}{dx} dx a_j &= \sum_{i=1}^m \sum_{j=1}^m a_i K_{ij} a_j = \mathbf{a}^T \mathbf{K} \mathbf{a} \geq 0 \end{aligned}$$

- **Absolute Minimum Property of Total Potential Energy ($u^h = u - u^e$)**

$$\begin{aligned}
 \Pi^h &= \frac{1}{2} \int_0^l \frac{du^h}{dx} \frac{du^h}{dx} dx - \int_0^l u^h f dx = \frac{1}{2} \int_0^l \frac{d(u - u^e)}{dx} \frac{d(u - u^e)}{dx} dx - \int_0^l (u - u^e) f dx \\
 &= \frac{1}{2} \int_0^l \frac{du}{dx} \frac{du}{dx} dx - \int_0^l u f dx + \frac{1}{2} \int_0^l \frac{du^e}{dx} \frac{du^e}{dx} dx - \int_0^l \frac{du^e}{dx} \frac{du}{dx} dx + \int_0^l u^e f dx \\
 &= \frac{1}{2} \int_0^l \frac{du}{dx} \frac{du}{dx} dx - \int_0^l u f dx + \frac{1}{2} \int_0^l \frac{du^e}{dx} \frac{du^e}{dx} dx - u^e \left. \frac{du}{dx} \right|_0^l + \int_0^l u^e \frac{d^2 u}{dx^2} dx + \int_0^l u^e f dx \\
 &= \frac{1}{2} \int_0^l \frac{du}{dx} \frac{du}{dx} dx - \int_0^l u f dx + \frac{1}{2} \int_0^l \frac{du^e}{dx} \frac{du^e}{dx} dx + \int_0^l u^e \left(\frac{d^2 u}{dx^2} + f \right) dx \\
 &= \Pi^E + \frac{1}{2} \int_0^l \left(\frac{du^e}{dx} \right)^2 dx \geq \Pi^E \quad (\text{The equality sign holds only for } u^e = C)
 \end{aligned}$$

- **Weighted Residual Method**

$$\pi_k = \int_0^l \phi_k R_E dx = \int_0^l \phi_k \left(\frac{d^2 u^h}{dx^2} + f \right) dx = 0 \quad \text{for } k = 1, \dots, m$$

- if $\phi_k = g_k$: Galerkin Method

$$\begin{aligned} \pi_k &= \int_0^l \phi_k \left(\frac{d^2 u^h}{dx^2} + f \right) dx = g_k \left. \frac{du^h}{dx} \right|_0^l - \int_0^l \frac{dg_k}{dx} \frac{du^h}{dx} dx + \int_0^l g_k f dx \\ &= - \int_0^l \frac{dg_k}{dx} \frac{du^h}{dx} dx + \int_0^l g_k f dx = - \int_0^l \frac{dg_k}{dx} \sum_{i=1}^m \frac{dg_i}{dx} a_i dx + \int_0^l g_k f dx \\ &= - \sum_{i=1}^m \int_0^l \frac{dg_k}{dx} \frac{dg_i}{dx} a_i dx + \int_0^l g_k f dx = 0 \quad \text{for } k = 1, \dots, m \rightarrow \mathbf{K}\mathbf{a} = \mathbf{F} \end{aligned}$$

- Identical result to the R.R. Method or Variational Principle
- if $\phi_k \neq g_k$: Petrov-Galerkin Method

- **Weighted Residual vs. Variational Principle**

$$\pi_k = 0 \text{ for } k = 1, \dots, m \Leftrightarrow \sum_{i=1}^m \pi_i \delta a_i = 0 \quad \forall \delta a_i$$

$$\sum_{i=1}^m \pi_i \delta a_i = \sum_{i=1}^m \delta a_i \int_0^l g_i \left(\frac{d^2 u^h}{dx^2} + f \right) dx = \int_0^l \sum_{i=1}^m \delta a_i g_i \left(\frac{d^2 u^h}{dx^2} + f \right) dx =$$

$$\int_0^l \delta u^h \left(\frac{d^2 u^h}{dx^2} + f \right) dx = - \left(\int_0^l \frac{d \delta u^h}{dx} \frac{du^h}{dx} dx - \int_0^l \delta u^h f dx \right) = 0 \quad \text{for all possible } \delta u^h$$

- **Example 1:** $\frac{d^2u}{dx^2} = -1, \quad u(0) = u(1) = 0$

- Exact solution : $u = -\frac{1}{2}x^2 + \frac{1}{2}x$

- First trial : $u^h = a_1 + a_2x + a_3x^2$

$$\text{Applying BCs : } a_1 = 0, \quad a_2 = -a_3 \rightarrow u^h = a_3(-x + x^2), \quad \frac{du^h}{dx} = a_3(-1 + 2x)$$

$$K_{11} = \int_0^1 (-1 + 2x)^2 dx = \frac{1}{3}, \quad F_1 = \int_0^1 1 \cdot (-x + x^2) dx = -\frac{1}{6}$$

$$\frac{1}{3}a_3 = -\frac{1}{6} \rightarrow a_3 = -\frac{1}{2}. \quad \text{Therefore, } u^h \equiv u$$

- Second trial: $u^h = a_1 + a_2x + a_3x^2 + a_4x^3$

$$\text{Applying BCs : } a_1 = 0, \quad a_2 = -a_3 - a_4 \rightarrow u^h = a_3 \underbrace{(-x + x^2)}_{g_1} + a_4 \underbrace{(-x + x^3)}_{g_2}$$

$$\frac{dg_1}{dx} = (-1 + 2x), \quad \frac{dg_2}{dx} = (-1 + 3x^2)$$

$$K_{11} = \int_0^1 (-1+2x)^2 dx = \frac{1}{3}, \quad K_{22} = \int_0^1 (-1+3x^2)^2 dx = \frac{4}{5}$$

$$K_{12} = K_{21} = \int_0^1 (-1+2x)(-1+3x^2) dx = \frac{1}{2},$$

$$F_1 = \int_0^1 1 \cdot (-x + x^2) dx = -\frac{1}{6}, \quad F_2 = \int_0^1 1 \cdot (-x + x^3) dx = -\frac{1}{4}$$

System Equation:

$$\begin{bmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{1}{2} & \frac{4}{5} \end{bmatrix} \begin{Bmatrix} a_3 \\ a_4 \end{Bmatrix} = \begin{Bmatrix} -\frac{1}{6} \\ -\frac{1}{4} \end{Bmatrix} \rightarrow a_3 = -\frac{1}{2}, a_4 = 0$$

- **Example 2:** $\frac{d^2u}{dx^2} = -\pi^2 \sin \pi x, u(0) = u(1) = 0$

- Exact Solution $u = \sin \pi x$

- First Trial $u^h = a_0 + a_1x + a_2x^2$

$$\text{Applying BC: } u^h = a_2(-x + x^2), \frac{du^h}{dx} = a_2(-1 + 2x)$$

$$K_{11} = \int_0^1 (-1 + 2x)^2 dx = \frac{1}{3}, F_1 = \pi^2 \int_0^1 \sin \pi x \cdot (-x + x^2) dx = -\frac{4}{\pi}$$

$$\frac{1}{3}a_2 = -\frac{4}{\pi} \rightarrow a_2 = -\frac{12}{\pi} \quad \text{Therefore, } u^h = -\frac{12}{\pi}(-x + x^2)$$

$$u^h(0.5) = \frac{3}{\pi} = 0.955 \quad \text{Error} = 4.5 \%$$

- Second trial: $u^h = a_0 + a_1x + a_2x^2 + a_3x^3$

$$\text{Applying BCs: } u^h = a_2 \underbrace{(-x + x^2)}_{g_1} + a_3 \underbrace{(-x + x^3)}_{g_2}$$

$$F_1 = \pi^2 \int_0^1 \sin \pi x \cdot (-x + x^2) dx = -\frac{4}{\pi}, \quad F_2 = \pi^2 \int_0^1 \sin \pi x \cdot (-x + x^3) dx = -\frac{6}{\pi}$$

System Equation: $\begin{bmatrix} 1 & 1 \\ \frac{3}{2} & 2 \\ \frac{1}{2} & 4 \\ \frac{1}{2} & 5 \end{bmatrix} \begin{Bmatrix} a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} -\frac{4}{\pi} \\ -\frac{6}{\pi} \end{Bmatrix} \rightarrow a_2 = -\frac{12}{\pi}, a_3 = 0 \quad \text{????}$

- In general: $u^h = \sum_{i=2}^m a_i (-x + x^i)$

$$\text{Function Error} = \left(\frac{\int_0^1 (\sin \pi x - u^h)^2 dx}{\int_0^1 \sin^2 \pi x dx} \right)^{1/2}, \quad \text{Derivative Error} = \left(\frac{\int_0^1 (\pi \cos \pi x - (u^h)')^2 dx}{\pi^2 \int_0^1 \cos^2 \pi x dx} \right)^{1/2}$$

To evaluate numerator in the error expressions, the midpoint rule with 100 subintervals is employed.

- Raw Output

******* 2th-order Polynomial *******

a 2 = -0.3819719E+01

Errors for 2th-order polynomial

Function error = 0.2009211E+01 %/Derivative error = 0.6010036E+01 %

******* 3th-order Polynomial *******

a 2 = -0.3819719E+01

a 3 = 0.0000000E+00

Errors for 3th-order polynomial

Function error = 0.2009211E+01 %/Derivative error = 0.6010036E+01 %

******* 4th-order Polynomial *******

a 2 = 0.4193832E+00

a 3 = -0.7065170E+01

a 4 = 0.3532585E+01

Errors for 4th-order polynomial

Function error = 0.4048880E-01 %/Derivative error = 0.1956108E+00 %

******* 5th-order Polynomial *******

```
a 2 = 0.4193832E+00  
a 3 = -0.7065170E+01  
a 4 = 0.3532585E+01  
a 5 = 0.7833734E-12
```

Errors for 5th-order polynomial

Function error = 0.4048880E-01 %/Derivative error = 0.1956108E+00 %

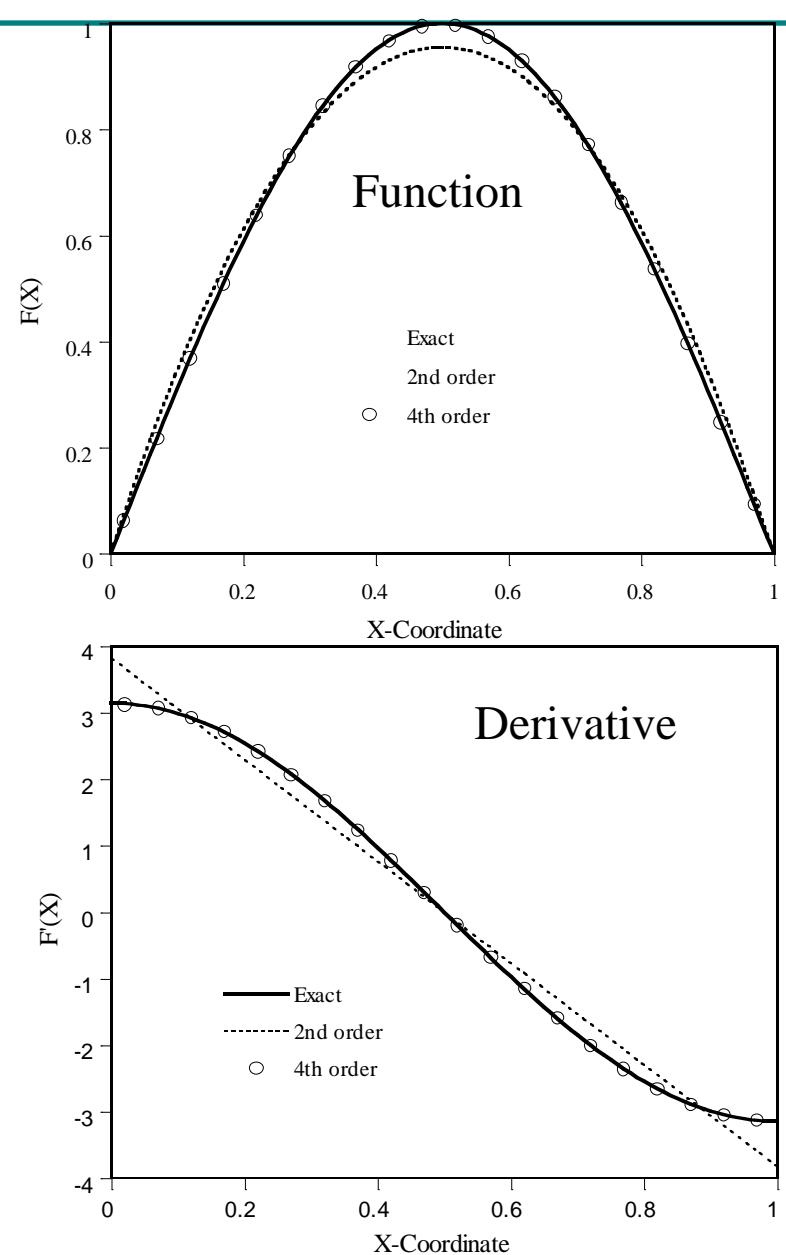
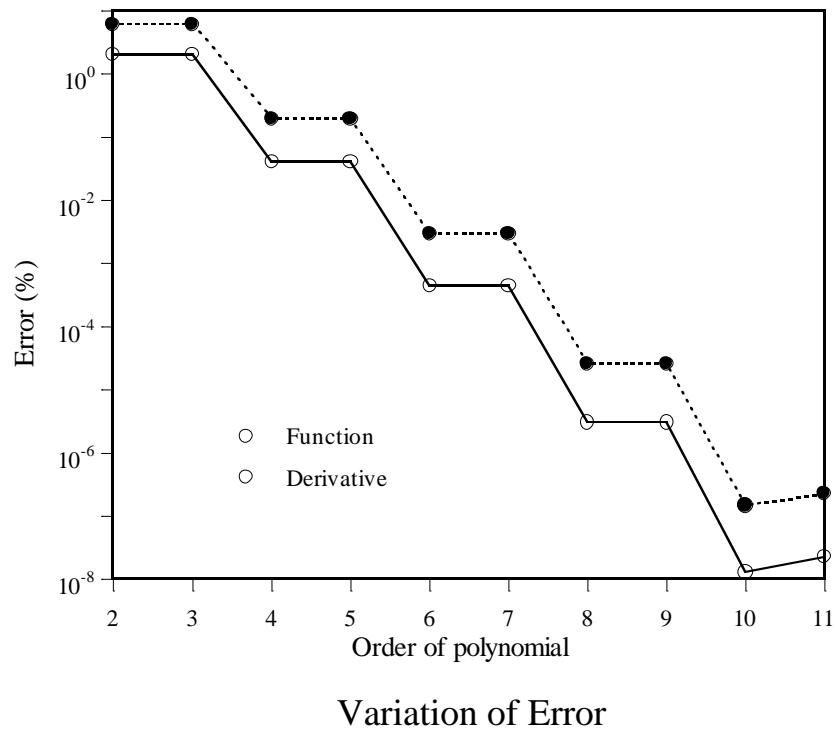
******* 6th-order Polynomial *******

```
a 2 = -0.1405727E-01  
a 3 = -0.5042448E+01  
a 4 = -0.5128593E+00  
a 5 = 0.3640900E+01  
a 6 = -0.1213633E+01
```

Errors for 6th-order polynomial

Function error = 0.4444114E-03 %/Derivative error = 0.2944068E-02 %

- Result plots



Homework 2

3.2. Problems with Traction Boundary Conditions

- **Problem Definition.**

- Differential Equation : $\frac{d^2u}{dx^2} + f = 0 \quad 0 < x < l$
- Boundary Conditions : at $x = 0 \ u = 0$ or $\frac{du}{dx} = \bar{T}$ and at $x = l \ u = 0$ or $\frac{du}{dx} = \bar{T}$

- **Error Minimization: Error estimator**

$$\begin{aligned}
 \Pi^R &= \frac{1}{2} \int_0^l (u - u^h) \left(\frac{d^2u^h}{dx^2} + f \right) dx + \frac{1}{2} (u - u^h) \left(\frac{du}{dx} - \frac{du^h}{dx} \right) \Big|_0^l \\
 &= \frac{1}{2} (u - u^h) \left(\frac{du^h}{dx} - \frac{du}{dx} \right) \Big|_0^l - \frac{1}{2} \int_0^l \left(\frac{du}{dx} - \frac{du^h}{dx} \right) \left(\frac{du^h}{dx} - \frac{du}{dx} \right) dx + \frac{1}{2} (u - u^h) \left(\frac{du}{dx} - \frac{du^h}{dx} \right) \Big|_0^l \\
 &= \frac{1}{2} \int_0^l \left(\frac{du}{dx} - \frac{du^h}{dx} \right) \left(\frac{du}{dx} - \frac{du^h}{dx} \right) dx = \Pi^{LS}
 \end{aligned}$$

- Energy Functional – Total Potential Energy

$$\begin{aligned}
 \Pi^R &= \frac{1}{2} \int_0^l (u - u^h) \left(\frac{d^2 u^h}{dx^2} + f \right) dx + \frac{1}{2} (u - u^h) \left(\frac{du}{dx} - \frac{du^h}{dx} \right) \Big|_0^l \\
 &= \frac{1}{2} \int_0^l \left(u \frac{d^2 u^h}{dx^2} + uf - u^h \frac{d^2 u^h}{dx^2} - u^h f \right) dx + \frac{1}{2} (u - u^h) \left(\frac{du}{dx} - \frac{du^h}{dx} \right) \Big|_0^l \\
 &= \frac{1}{2} \int_0^l (uf - u^h f) dx + \frac{1}{2} \left\{ u \frac{du^h}{dx} \Big|_0^l - \frac{du}{dx} u^h \Big|_0^l + \int_0^l \frac{d^2 u}{dx^2} u^h dx - u^h \frac{du^h}{dx} \Big|_0^l + \int_0^l \frac{du^h}{dx} \frac{du^h}{dx} dx \right\} \\
 &\quad + \frac{1}{2} \left(u \frac{du}{dx} - u \frac{du^h}{dx} - u^h \frac{du}{dx} + u^h \frac{du^h}{dx} \right) \Big|_0^l \\
 &= \frac{1}{2} \left(\int_0^l u f dx + u \bar{T} \Big|_0^l \right) + \frac{1}{2} \int_0^l \frac{du^h}{dx} \frac{du^h}{dx} dx - \int_0^l u^h f dx - u^h \bar{T} \Big|_0^l \\
 &= C + \Pi^{RR}
 \end{aligned}$$

- Minimization Problems

$$\text{Min } \Pi^R \Leftrightarrow \text{Min } \Pi^{LS} \Leftrightarrow \text{Min } \Pi^{RR} \text{ w.r.t. } u^h \in \mathcal{V}^h$$

- Min Π^{RR} : **Rayleigh-Ritz Method** or **Principle of Minimum Potential Energy**

- 1st Order Necessary Condition of Minimization Problem

$$\begin{aligned}
 \frac{\partial \Pi^{RR}}{\partial a_k} &= \int_0^l \frac{d}{da_k} \left(\frac{du^h}{dx} \right) \frac{du^h}{dx} dx - \int_0^l \frac{du^h}{da_k} f dx - \frac{du^h}{da_k} \bar{T} \Big|_0^l \\
 &= \int_0^l \frac{d}{da_k} \left(\sum_{i=1}^m a_i \frac{dg_i}{dx} \right) \left(\sum_{i=1}^m a_i \frac{dg_i}{dx} \right) dx - \int_0^l \frac{d}{da_k} \left(\sum_{i=1}^m a_i g_i \right) f dx - \frac{d}{da_k} \sum_{i=1}^m a_i g_i \bar{T} \Big|_0^l \\
 &= \sum_{i=1}^m \int_0^l \frac{dg_k}{dx} \frac{dg_i}{dx} dx a_i - \int_0^l g_k f dx - g_k \bar{T} \Big|_0^l = \sum_{i=1}^{m'} K_{ki} a_i - F_k = 0 \quad \text{for } i = 1, \dots, m \rightarrow \mathbf{Ka} = \mathbf{F}
 \end{aligned}$$

- $\delta \Pi^{RR} = 0$: **Variational Principle** or **Principle of Virtual Work**

$$\begin{aligned}
 \delta \Pi^{RR} &= \delta \left(\frac{1}{2} \int_0^l \frac{du^h}{dx} \frac{du^h}{dx} dx - \int_0^l u^h f dx - u^h \bar{T} \Big|_0^l \right) = \int_0^l \frac{d\delta u^h}{dx} \frac{du^h}{dx} dx - \int_0^l \delta u^h f dx - \delta u^h \bar{T} \Big|_0^l = \\
 &= \sum_{k=1}^m \delta a_k \left(\sum_{i=1}^m \int_0^l \frac{dg_k}{dx} \frac{dg_i}{dx} dx a_i - \int_0^l g_k f dx - g_k \bar{T} \Big|_0^l \right) \\
 &= \sum_{k=1}^m \delta a_k \left(\sum_{i=1}^m K_{ki} a_i - F_k \right) = 0 \rightarrow (\delta \mathbf{a})^T (\mathbf{Ka} - \mathbf{F}) = 0
 \end{aligned}$$

- **Absolute Minimum Property of Total Potential Energy ($u^h = u - u^e$)**

$$\begin{aligned}
 \Pi^h &= \frac{1}{2} \int_0^l \frac{du^h}{dx} \frac{du^h}{dx} dx - \left[u^h f dx - u^h \bar{T} \right]_0^l \\
 &= \frac{1}{2} \int_0^l \frac{d(u - u^e)}{dx} \frac{d(u - u^e)}{dx} dx - \left[(u - u^e) f dx - (u - u^e) \bar{T} \right]_0^l \\
 &= \frac{1}{2} \int_0^l \frac{du}{dx} \frac{du}{dx} dx - \left[u f dx - u \bar{T} \right]_0^l + \frac{1}{2} \int_0^l \frac{du^e}{dx} \frac{du^e}{dx} dx - \left[\frac{du^e}{dx} \frac{du}{dx} dx + \int_0^l u^e f dx + u^e \bar{T} \right]_0^l \\
 &= \frac{1}{2} \int_0^l \frac{du}{dx} \frac{du}{dx} dx - \left[u f dx - u \bar{T} \right]_0^l + \frac{1}{2} \int_0^l \frac{du^e}{dx} \frac{du^e}{dx} dx - \left[u^e \frac{du}{dx} \Big|_0^l + \int_0^l u^e \frac{d^2 u}{dx^2} dx + \int_0^l u^e f dx + u^e \bar{T} \right]_0^l \\
 &= \frac{1}{2} \int_0^l \frac{du}{dx} \frac{du}{dx} dx - \left[u f dx - u \bar{T} \right]_0^l + \frac{1}{2} \int_0^l \frac{du^e}{dx} \frac{du^e}{dx} dx + \left[u^e \left(\frac{d^2 u}{dx^2} + f \right) dx + u^e \left(\bar{T} - \frac{du}{dx} \right) \Big|_0^l \right]_0^l \\
 &= \frac{1}{2} \int_0^l \frac{du}{dx} \frac{du}{dx} dx - \left[u f dx - u \bar{T} \right]_0^l + \frac{1}{2} \int_0^l \frac{du^e}{dx} \frac{du^e}{dx} dx \\
 &= \Pi^E + \frac{1}{2} \int_0^l \left(\frac{du^e}{dx} \right)^2 dx \geq \Pi^E \quad (\text{The equality sign holds only for } u^e = C)
 \end{aligned}$$

- **Weighted Residual Method**

$$\begin{aligned}
 \pi_k &= \int_0^l g_k R_E dx + g_k \left(\frac{du}{dx} - \frac{du^h}{dx} \right) \Big|_0^l = \int_0^l g_k \left(\frac{d^2 u^h}{dx^2} + f \right) dx + g_k \left(\frac{du}{dx} - \frac{du^h}{dx} \right) \Big|_0^l \\
 &= g_k \frac{du^h}{dx} \Big|_0^l - \int_0^l \frac{dg_k}{dx} \frac{du^h}{dx} dx + \int_0^l g_k f dx + g_k \left(\frac{du}{dx} - \frac{du^h}{dx} \right) \Big|_0^l = - \int_0^l \frac{dg_k}{dx} \frac{du^h}{dx} dx + \int_0^l g_k f dx + g_k \bar{T} \Big|_0^l \\
 &= - \sum_{i=1}^m \int_0^l \frac{dg_k}{dx} \frac{dg_i}{dx} dx a_i + \int_0^l g_k f dx + g_k \bar{T} \Big|_0^l = 0 \quad \text{for } k = 1, \dots, m \rightarrow \mathbf{K}\mathbf{a} = \mathbf{F}
 \end{aligned}$$

- **Weighted Residual vs. Variational Principle**

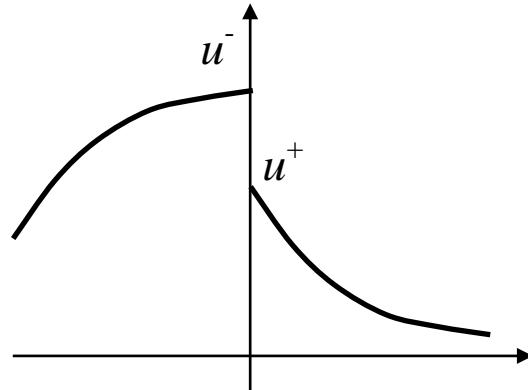
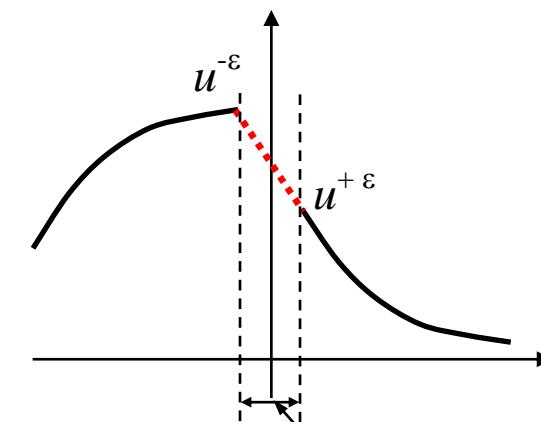
$$\pi_k = 0 \text{ for } k = 1, \dots, m \Leftrightarrow \sum_{i=1}^m \pi_i \delta a_i = 0 \text{ for all possible } \delta a_i$$

$$\begin{aligned}
 \sum_{k=1}^m \delta a_k \left(- \int_0^l \frac{dg_k}{dx} \frac{du^h}{dx} dx + \int_0^l g_k f dx + g_k \bar{T} \Big|_0^l \right) &= \sum_{k=1}^m \left(- \int_0^l \frac{d\delta a_k}{dx} g_k \frac{du^h}{dx} dx + \int_0^l \delta a_k g_k f dx + \delta a_k g_k \bar{T} \Big|_0^l \right) = \\
 - \int_0^l \frac{d\delta u^h}{dx} \frac{du^h}{dx} dx + \int_0^l \delta u^h f dx + \delta u^h \bar{T} \Big|_0^l &= \delta \Pi^{RR} = 0
 \end{aligned}$$

3.3 Integrability Condition

– Regularity (Continuity) Requirement –

- Integration of functions with discontinuity $\int_{-l}^l f(u(x))dx \quad ??$

Original function u Function with transition zone \bar{u}

$$\int_{-l}^l f(u(x))dx = \lim_{\varepsilon \rightarrow 0} \int_{-l}^l f(\bar{u}(x))dx$$

where $\bar{u} = \begin{cases} u & \text{for } -l \leq x \leq -\varepsilon \\ \frac{u^{+\varepsilon} - u^{-\varepsilon}}{2\varepsilon} x + \frac{u^{+\varepsilon} + u^{-\varepsilon}}{2} & \text{for } -\varepsilon \leq x \leq \varepsilon \text{ and } u^{+\varepsilon} = u(\varepsilon), u^{-\varepsilon} = u(-\varepsilon) \\ u & \text{for } \varepsilon \leq x \leq l \end{cases}$

- Can we integrate $\int_{-l}^l u dx$ on what condition?

$$\int_{-l}^l \bar{u} dx = \int_{-l}^{-\varepsilon} \bar{u} dx + \int_{-\varepsilon}^{\varepsilon} \bar{u} dx + \int_{\varepsilon}^l \bar{u} dx$$

$$\int_{-\varepsilon}^{\varepsilon} \bar{u} dx = \int_{-\varepsilon}^{\varepsilon} \left(\frac{u^{+\varepsilon} - u^{-\varepsilon}}{2\varepsilon} x + \frac{u^{+\varepsilon} + u^{-\varepsilon}}{2} \right) dx = (u^{+\varepsilon} + u^{-\varepsilon})\varepsilon$$

$$\int_{-l}^l u dx = \lim_{\varepsilon \rightarrow 0} \left(\int_{-l}^{-\varepsilon} \bar{u} dx + \int_{-\varepsilon}^{\varepsilon} \bar{u} dx + \int_{\varepsilon}^l \bar{u} dx \right) = \lim_{\varepsilon \rightarrow 0} \left(\int_{-l}^{-\varepsilon} \bar{u} dx + \int_{\varepsilon}^l \bar{u} dx + (u^{+\varepsilon} + u^{-\varepsilon})\varepsilon \right)$$

The last integral vanishes as far as u^- and u^+ are finite, and the integral becomes

$$\int_{-l}^l u dx = \int_{-l}^0 u dx + \int_0^l u dx$$

- Can we integrate $\int_{-l}^l u^2 dx$ on what condition ?

$$\int_{-l}^l \bar{u}^2 dx = \int_{-l}^{-\varepsilon} \bar{u}^2 dx + \int_{-\varepsilon}^{\varepsilon} \bar{u}^2 dx + \int_{\varepsilon}^l \bar{u}^2 dx$$

$$\int_{-\varepsilon}^{\varepsilon} \bar{u}^2 dx = \int_{-\varepsilon}^{\varepsilon} \left(\frac{u^{+\varepsilon} - u^{-\varepsilon}}{2\varepsilon} x + \frac{u^{+\varepsilon} + u^{-\varepsilon}}{2} \right)^2 dx = (u^{+\varepsilon} - u^{-\varepsilon})^2 \frac{\varepsilon}{6} + (u^{+\varepsilon} + u^{-\varepsilon})^2 \frac{\varepsilon}{2}$$

$$\int_{-l}^l u^2 dx = \lim_{\varepsilon \rightarrow 0} \left(\int_{-l}^{-\varepsilon} \bar{u}^2 dx + \int_{-\varepsilon}^{\varepsilon} \bar{u}^2 dx + \int_{\varepsilon}^l \bar{u}^2 dx \right) = \lim_{\varepsilon \rightarrow 0} \left(\int_{-l}^{-\varepsilon} \bar{u}^2 dx + (u^{+\varepsilon} - u^{-\varepsilon})^2 \frac{\varepsilon}{6} + (u^{+\varepsilon} + u^{-\varepsilon})^2 \frac{\varepsilon}{2} \right)$$

The last integral vanishes as far as u^- and u^+ are finite, and the integral becomes

$$\int_{-l}^l u^2 dx = \int_{-l}^0 u^2 dx + \int_0^l u^2 dx$$

- Can we integrate $\int_{-l}^l \left(\frac{du}{dx}\right)^2 dx$??

$$\begin{aligned} \int_{-l}^l \left(\frac{d\bar{u}}{dx}\right)^2 dx &= \int_{-l}^{-\varepsilon} \left(\frac{d\bar{u}}{dx}\right)^2 dx + \int_{-\varepsilon}^{\varepsilon} \left(\frac{d\bar{u}}{dx}\right)^2 dx + \int_{\varepsilon}^l \left(\frac{d\bar{u}}{dx}\right)^2 dx = \int_{-l}^{-\varepsilon} \left(\frac{d\bar{u}}{dx}\right)^2 dx + \int_{-\varepsilon}^{\varepsilon} \left(\frac{u^{+\varepsilon} - u^{-\varepsilon}}{2\varepsilon}\right)^2 dx + \int_{\varepsilon}^l \left(\frac{d\bar{u}}{dx}\right)^2 dx \\ &= \int_l^{-\varepsilon} \left(\frac{d\bar{u}}{dx}\right)^2 dx + \frac{(u^{+\varepsilon} - u^{-\varepsilon})^2}{2\varepsilon} + \int_{\varepsilon}^l \left(\frac{d\bar{u}}{dx}\right)^2 dx \end{aligned}$$

$$\int_{-l}^l \left(\frac{du}{dx}\right)^2 dx = \lim_{\varepsilon \rightarrow 0} \int_{-l}^{-\varepsilon} \left(\frac{d\bar{u}}{dx}\right)^2 dx = \int_{-l}^0 \left(\frac{du}{dx}\right)^2 dx + \int_0^l \left(\frac{du}{dx}\right)^2 dx + \lim_{\varepsilon \rightarrow 0} \frac{(u^{+\varepsilon} - u^{-\varepsilon})^2}{2\varepsilon}$$

Therefore, the given definite integral has a finite value if and only if u is continuous.

$$\lim_{\varepsilon \rightarrow 0} u(+\varepsilon) = \lim_{\varepsilon \rightarrow 0} u(-\varepsilon)$$

From the physical point of view, the aforementioned continuity condition represents the **compatibility condition**, which states that the displacement field in a continuum should be uniquely determined, ie, defined by single valued function.

- Can we integrate $\int_{-l}^l u \frac{du}{dx} dx$ on what condition ?

$$\int_{-l}^l \bar{u} \frac{d\bar{u}}{dx} dx = \int_{-l}^{-\varepsilon} \bar{u} \frac{d\bar{u}}{dx} dx + \int_{-\varepsilon}^{\varepsilon} \bar{u} \frac{d\bar{u}}{dx} dx + \int_{\varepsilon}^l \bar{u} \frac{d\bar{u}}{dx} dx$$

$$\int_{-\varepsilon}^{\varepsilon} \bar{u} \frac{d\bar{u}}{dx} dx = \int_{-\varepsilon}^{\varepsilon} \left(\frac{u^{+\varepsilon} - u^{-\varepsilon}}{2\varepsilon} x + \frac{u^{+\varepsilon} + u^{-\varepsilon}}{2} \right) \left(\frac{u^{+\varepsilon} - u^{-\varepsilon}}{2\varepsilon} \right) dx = (u^{+\varepsilon} - u^{-\varepsilon}) \frac{(u^{+\varepsilon} + u^{-\varepsilon})}{2} \text{ or}$$

$$\int_{-\varepsilon}^{\varepsilon} \bar{u} \frac{d\bar{u}}{dx} dx = \bar{u}^2 \Big|_{-\varepsilon}^{\varepsilon} - \int_{-\varepsilon}^{\varepsilon} \frac{d\bar{u}}{dx} \bar{u} dx \rightarrow \int_{-\varepsilon}^{\varepsilon} \frac{d\bar{u}}{dx} \bar{u} dx = \frac{1}{2} (u^2(\varepsilon) - u^2(-\varepsilon)) = (u^{+\varepsilon} - u^{-\varepsilon}) \frac{(u^{+\varepsilon} + u^{-\varepsilon})}{2}$$

$$\int_{-l}^l u \frac{du}{dx} dx = \lim_{\varepsilon \rightarrow 0} \left(\int_{-l}^{-\varepsilon} \bar{u} \frac{d\bar{u}}{dx} dx + \int_{-\varepsilon}^{\varepsilon} \bar{u} \frac{d\bar{u}}{dx} dx + \int_{\varepsilon}^l \bar{u} \frac{d\bar{u}}{dx} dx \right)$$

$$= \lim_{\varepsilon \rightarrow 0} \left(\int_{-l}^{-\varepsilon} \bar{u} \frac{d\bar{u}}{dx} dx + \int_{\varepsilon}^l \bar{u} \frac{d\bar{u}}{dx} dx + (u^{+\varepsilon} - u^{-\varepsilon}) \frac{(u^{+\varepsilon} + u^{-\varepsilon})}{2} \right)$$

$$= \int_{-l}^0 u \frac{du}{dx} dx + \int_0^l u \frac{du}{dx} dx + (u^+ - u^-) \frac{(u^+ + u^-)}{2} = \int_{-l}^0 u \frac{du}{dx} dx + \int_0^l u \frac{du}{dx} dx + [u]^\pm(u)^\pm$$

Therefore, the given definite integral has a unique & finite value even if u is discontinuous.

Homework 3

3.4. The other side of Virtual Work

- **Physical Viewpoint**

If a deformable body is in equilibrium under a Q -force system and remains in equilibrium while it is subjected a small virtual deformation, the external virtual work done by external Q forces acting on the body is equal to the internal virtual work of deformation done by the internal Q -stresses.

$$\int_S \delta u_i Q_i dS = \int_V \delta \varepsilon_{ij} \sigma_{ij} dV \quad \text{where} \quad \delta \varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial \delta u_i}{\partial x_j} + \frac{\partial \delta u_j}{\partial x_i} \right)$$

- **Mathematical Viewpoint - Continuous Problem**

If $\mathbf{A}(\mathbf{u}) = 0$ should hold for a given system, then the following statement should hold. Here, $\mathbf{u} \in \mathbf{v}$, and the order of \mathbf{A} should be the same as \mathbf{u} .

$$\int_V \delta \mathbf{u} \cdot \mathbf{A}(\mathbf{u}) dV = 0 \quad \forall \delta \mathbf{u} \in \mathbf{v} \quad (\mathbf{v}: \text{A proper Function space})$$

Example : Beam problem

$$EI \frac{d^4 w}{dx^4} - q = 0$$

$$\int_V \delta w (EI \frac{d^4 w}{dx^4} - q) dx = 0 \rightarrow \int_V \frac{d^2 \delta w}{dx^2} EI \frac{d^2 w}{dx^2} dx = \int_V \delta w q dx$$

If δw is the displacement induced by the unit load applied load at x_j , the expression for the principle of virtual work becomes as follows.

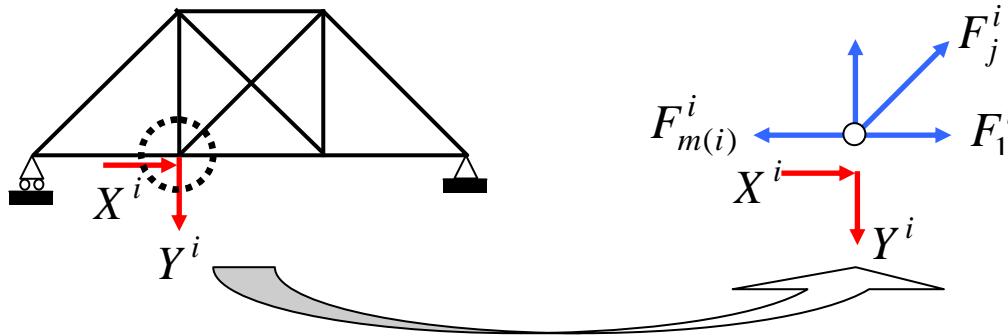
$$\int_v \frac{M^u M^Q}{EI} dx = \int_v \delta w q dx = \int_v w \delta(x - x_j) dx = w(x_j)$$

where M^u is the moment induced by the unit load applied at x_j and $\delta(x - x_j)$ is a delac delta function applied at x_j .

- **Mathematical Viewpoint - Discrete Problem**

$$\delta \mathbf{u} \cdot \mathbf{A}(\mathbf{u}) = \delta u_i \cdot A_i(\mathbf{u}) = 0 \quad \forall \delta \mathbf{u} \in \mathcal{V} \quad (\mathcal{V}: \text{A proper vector space})$$

- Example : Truss problem



- Equilibrium Equations at joints

$$\sum_{j=1}^{m(i)} H_j^i + X^i = 0 , \quad \sum_{j=1}^{m(i)} V_j^i + Y^i = 0 \quad \text{for } i = 1, \dots, n$$

- Virtual Work Expression

$$\begin{aligned} & \sum_{i=1}^n \left(\left(\sum_{j=1}^{m(i)} H_j^i + X^i \right) \delta u^i + \left(\sum_{j=1}^{m(i)} V_j^i + Y^i \right) \delta v^i \right) = 0 \\ & \sum_{i=1}^n \left(\left(- \sum_{j=1}^{m(i)} F_j^i \cos \theta_j^i + X^i \right) \delta u^i + \left(- \sum_{j=1}^{m(i)} F_j^i \sin \theta_j^i + Y^i \right) \delta v^i \right) = 0 \\ & \sum_{i=1}^n \left(\delta u^i \sum_{j=1}^{m(i)} F_j^i \cos \theta_j^i + \delta v^i \sum_{j=1}^{m(i)} F_j^i \sin \theta_j^i \right) = \sum_{i=1}^n (X^i \delta u^i + Y^i \delta v^i) \end{aligned}$$

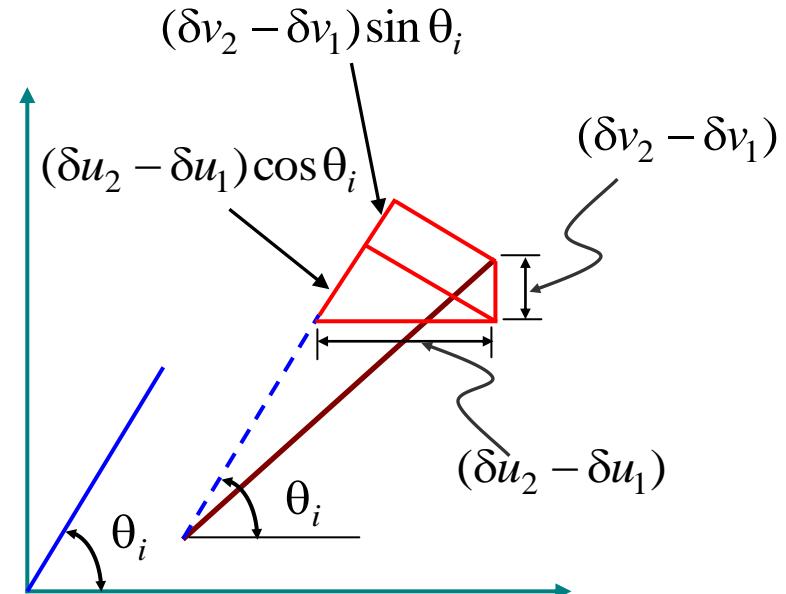
$$\sum_{i=1}^{nmb} (F^i \cos \theta_i (\delta u_2^i - \delta u_1^i) + F^i \sin \theta_i (\delta v_2^i - \delta v_1^i)) =$$

$$\sum_{i=1}^{nmb} F^i ((\delta u_2^i - \delta u_1^i) \cos \theta_i + (\delta v_2^i - \delta v_1^i) \sin \theta_i) =$$

$$\sum_{i=1}^{nmb} F^i \Delta l_\mu^i = \sum_{i=1}^{nmb} F^i \frac{\mu^i l^i}{(EA^i)} = \sum_{i=1}^n (X^i \delta u^i + Y^i \delta v^i) = \sum_{i=1}^n (X_\mu^i u^i + Y_\mu^i v^i)$$

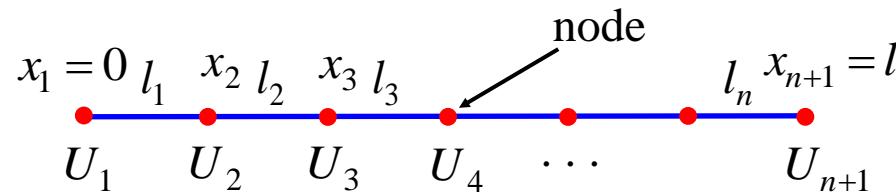
If μ force system consists of an unit load applied at k-th joint in arbitrary direction, then

$$X_\mu^k u^k + Y_\mu^k v^k = \| \mathbf{X}_\mu \| \| \mathbf{u} \| \cos \theta = \| \mathbf{u} \| \cos \theta = \sum_{i=1}^{nmb} \frac{F^i \mu^i l^i}{(EA^i)}$$



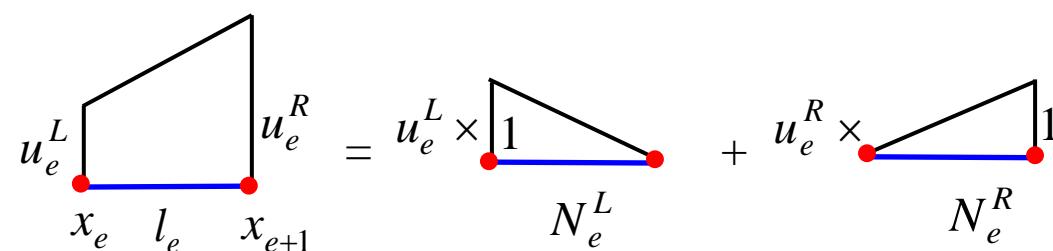
3.5. Finite Element Discretization

- Domain Discretization



$$\begin{aligned}\delta\Pi &= \int_0^l \frac{d\delta u^h}{dx} \frac{du^h}{dx} dx - \int_0^l f \delta u^h dx - \delta u^h \bar{T} \Big|_0^l = \sum_{e=1}^n \int_{l_e}^{l_e} \frac{d\delta u^h}{dx} \frac{du^h}{dx} dx - \sum_{e=1}^n \int_{l_e}^{l_e} f \delta u^h dx - \delta u^h \bar{T} \Big|_0^l \\ &= \sum_e \int_{l_e}^{l_e} \frac{d\delta u^h}{dx} \frac{du^h}{dx} dx - \sum_e \int_{l_e}^{l_e} f \delta u^h dx - \delta u^h \bar{T} \Big|_0^l = 0 \quad \text{for all admissible } \delta u^h\end{aligned}$$

- Interpolation of the Displacement Field in an Element.



$$u_e^h(x) = N_e^L u_e^L + N_e^R u_e^R = (N_e^L, N_e^R) \begin{pmatrix} u_e^L \\ u_e^R \end{pmatrix} = \mathbf{N} \cdot \mathbf{u}_e \quad , \quad \delta u_e^h(x) = \mathbf{N} \cdot \delta \mathbf{u}_e$$

$$N_e^L = \frac{x_{e+1} - x}{x_{e+1} - x_e} = \frac{x_{e+1} - x}{l_e}, \quad N_e^R = \frac{x - x_e}{x_{e+1} - x_e} = \frac{x - x_e}{l_e}$$

- **Discretized Form of Variational Statement**

$$\frac{du_e^h}{dx} = \frac{dN_e^L}{dx} u_e^L + \frac{dN_e^R}{dx} u_e^R = \left(\frac{dN_e^L}{dx}, \frac{dN_e^R}{dx} \right) \begin{pmatrix} u_e^L \\ u_e^R \end{pmatrix} = \mathbf{B} \cdot \mathbf{u}_e$$

$$\frac{d\delta u_e^h}{dx} = \frac{dN_e^L}{dx} \delta u_e^L + \frac{dN_e^R}{dx} \delta u_e^R = \mathbf{B} \cdot \begin{pmatrix} \delta u_e^L \\ \delta u_e^R \end{pmatrix} = \mathbf{B} \cdot \delta \mathbf{u}_e$$

$$\begin{aligned} \delta \Pi &= \sum_e \int_{l_e} \frac{d\delta u^h}{dx} \frac{du^h}{dx} dx - \sum_e \int_{l_e} f \delta u^h dx - \delta u^h \bar{T} \Big|_0^l \\ &= \sum_e \int_{l_e} \left(\frac{d\delta u_e^h}{dx} \right)^T \frac{du_e^h}{dx} dx - \sum_e \int_{l_e} (\delta u_e^h)^T f dx - (\delta u_n^R \bar{T}(l) - \delta u_1^L(0) \bar{T}(0)) \\ &= \sum_e \delta \mathbf{u}_e^T \int_{l_e} \mathbf{B}^T \mathbf{B} dx \mathbf{u}_e - \sum_e \delta \mathbf{u}_e^T \int_{l_e} \mathbf{N}^T f dx - \delta \mathbf{u}^b \bar{\mathbf{T}} = \sum_e \delta \mathbf{u}_e^T \mathbf{K}_e \mathbf{u}_e - \sum_e \delta \mathbf{u}_e^T \mathbf{F}_e - \delta \mathbf{u}^b \bar{\mathbf{T}} = 0 \end{aligned}$$

- **Compatibility Condition – Continuity Requirement**

$$u_e^L = u_{e-1}^R = U^e, u_e^R = u_{e+1}^L = U^{e+1}$$

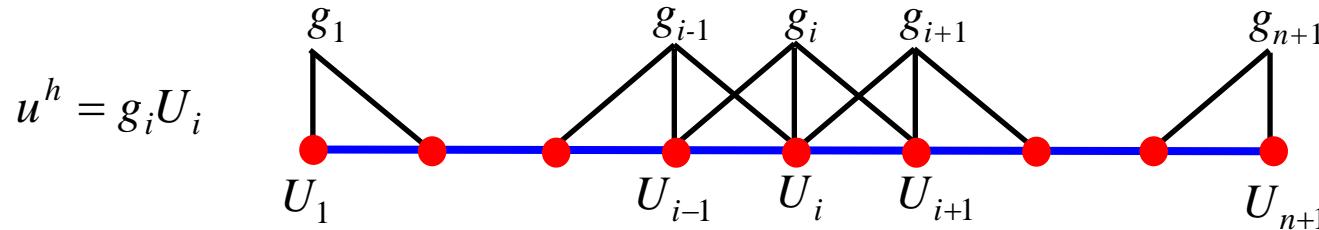
$$\mathbf{u}_e = \begin{pmatrix} u_e^L \\ u_e^R \end{pmatrix} = \begin{bmatrix} 0 & \dots & 1 & 0 & \dots & 0 \\ 0 & \dots & 0 & 1 & \dots & 0 \end{bmatrix} \begin{pmatrix} U^1 \\ \vdots \\ U^e \\ U^{e+1} \\ \vdots \\ U^{n+1} \end{pmatrix} = \mathbf{C}_e \mathbf{U}, \quad \delta \mathbf{u}_e = \mathbf{C}_e \delta \mathbf{U}$$

- **Global System Equation – Global Stiffness Equation**

$$\begin{aligned} \delta \Pi &= \sum_e \delta \mathbf{U}^T \mathbf{C}_e^T \mathbf{K}_e \mathbf{C}_e \mathbf{U} - \sum_e \delta \mathbf{U}^T \mathbf{C}_e^T \mathbf{F}_e - \delta \mathbf{U}^T \mathbf{C}_b^T \bar{\mathbf{T}} = \delta \mathbf{U}^T (\sum_e \mathbf{C}_e^T \mathbf{K}_e \mathbf{C}_e) \mathbf{U} - \delta \mathbf{U}^T (\sum_e \mathbf{C}_e^T \mathbf{F}_e - \mathbf{C}_b^T \bar{\mathbf{T}}) \\ &= \delta \mathbf{U}^T (\mathbf{K} \mathbf{U} - \mathbf{F}) = 0 \quad \text{for all admissible } \delta \mathbf{U} \end{aligned}$$

KU = F

- Rayleigh-Ritz Interpretation of FEM



$$\int_0^l \frac{d\delta u^h}{dx} \frac{du^h}{dx} dx = \sum_{i=1}^{n+1} \delta U_i \sum_{j=1}^{n+1} \int_0^l \frac{dg_i}{dx} \frac{dg_j}{dx} dx U_j = \delta U_1 \int_{x_1}^{x_2} \frac{dg_1}{dx} \left(\frac{dg_1}{dx} U_1 + \frac{dg_2}{dx} U_2 \right) dx + \dots$$

$$\delta U_{i-1} \int_{x_{i-2}}^{x_{i-1}} \frac{dg_{i-1}}{dx} \left(\frac{dg_{i-2}}{dx} U_{i-2} + \frac{dg_{i-1}}{dx} U_{i-1} \right) dx + \delta U_{i-1} \int_{x_{i-1}}^{x_i} \frac{dg_{i-1}}{dx} \left(\frac{dg_{i-1}}{dx} U_{i-1} + \frac{dg_i}{dx} U_i \right) dx +$$

$$\delta U_i \int_{x_{i-1}}^{x_i} \frac{dg_i}{dx} \left(\frac{dg_{i-1}}{dx} U_{i-1} + \frac{dg_i}{dx} U_i \right) dx + \delta U_i \int_{x_i}^{x_{i+1}} \frac{dg_i}{dx} \left(\frac{dg_i}{dx} U_i + \frac{dg_{i+1}}{dx} U_{i+1} \right) dx + \dots$$

$$\delta U_{n+1} \int_{x_n}^{x_{n+1}} \frac{dg_{n+1}}{dx} \left(\frac{dg_n}{dx} U_n + \frac{dg_{n+1}}{dx} U_{n+1} \right) dx$$

$$= \sum_{i=1}^n \int_{x_i}^{x_{i+1}} (\delta U_i \frac{dg_i}{dx} + \delta U_{i+1} \frac{dg_{i+1}}{dx}) \left(\frac{dg_i}{dx} U_i + \frac{dg_{i+1}}{dx} U_{i+1} \right) dx = \sum_e \delta \mathbf{U}_e^T \int_{x_i}^{x_{i+1}} \mathbf{B}^T \mathbf{B} dx \mathbf{U}_e$$

- **Finite Element Procedure**

1. Governing equations in the domain, boundary conditions on the boundary.
2. Derive weak form of the G.E. and B.C. by the variational principle or equivalent.
3. Descretize the given domain and boundary with finite elements.

$$V = V^1 \cup V^2 \cup \dots \cup V^n, \quad S = S^1 \cup S^2 \cup \dots \cup S^m$$

4. Assume the displacement field by shape functions and nodal values within an element.

$$\mathbf{u}^e = \mathbf{N}^e \mathbf{U}^e$$

5. Calculate the element stiffness matrix and assemble it according to the compatibility.

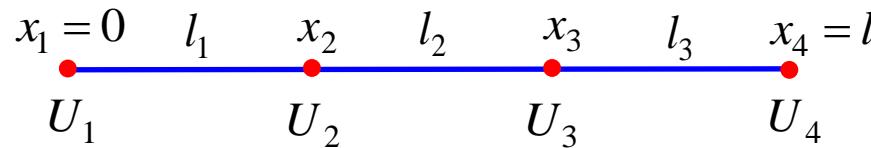
$$\mathbf{K}^e = \int_{l^e} \mathbf{B}^T \mathbf{D} \mathbf{B} dV, \quad \mathbf{K} = \sum_e \mathbf{K}^e$$

6. Calculate the equivalent nodal force and assemble it according to the compatibility.

$$\mathbf{F}^e = \int_{l^e} \mathbf{N}^T f dV + \mathbf{N}^T \bar{\mathbf{T}} \Big|_0^l, \quad \mathbf{F} = \sum_e \mathbf{F}^e$$

7. Apply the displacement boundary conditions and solve the stiffness equation.
8. Calculate strain, stress and reaction force.

- Example – with three elements and four nodes



- Shape Function Matrix

$$u_e^h(x) = N_e^L u_e^L + N_e^R u_e^R = (N_e^L, N_e^R) \begin{pmatrix} u_e^L \\ u_e^R \end{pmatrix} = \mathbf{N}_e \cdot \mathbf{u}_e$$

$$N_e^L = \frac{x_{e+1} - x}{x_{e+1} - x_e} = \frac{x_{e+1} - x}{l_e}, \quad N_e^R = \frac{x - x_e}{x_{e+1} - x_e} = \frac{x - x_e}{l_e}$$

$$\mathbf{B}_e = \left(\frac{dN_e^L}{dx}, \frac{dN_e^R}{dx} \right) = \frac{1}{l_e} [-1, 1]$$

- Element Stiffness Matrix

$$[\mathbf{K}]_e = \int_{V_e} \frac{1}{l_e} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \frac{1}{l_e} [-1, 1] dx = \frac{1}{l_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

- Compatibility Matrix

$$\mathbf{u}_1 = \begin{pmatrix} u_1^L \\ u_1^R \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{pmatrix} = \mathbf{C}_1 \mathbf{U}, \quad \mathbf{u}_2 = \begin{pmatrix} u_2^L \\ u_2^R \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{pmatrix} = \mathbf{C}_2 \mathbf{U}$$

$$\mathbf{u}_3 = \begin{pmatrix} u_3^L \\ u_3^R \end{pmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{pmatrix} = \mathbf{C}_3 \mathbf{U}$$

- Global Stiffness Matrix

$$\mathbf{K} = \sum_e \mathbf{C}_e^T \mathbf{K}_e \mathbf{C}_e = \mathbf{C}_1^T \mathbf{K}_1 \mathbf{C}_1 + \mathbf{C}_2^T \mathbf{K}_2 \mathbf{C}_2 + \mathbf{C}_3^T \mathbf{K}_3 \mathbf{C}_3 = \frac{1}{l_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned}
 & + \frac{1}{l_2} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} + \frac{1}{l_3} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
 & = \frac{1}{l_1} \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{1}{l_2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{1}{l_3} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \\
 & = \begin{bmatrix} \frac{1}{l_1} & -\frac{1}{l_1} & 0 & 0 \\ -\frac{1}{l_1} & \frac{1}{l_1} + \frac{1}{l_2} & -\frac{1}{l_2} & 0 \\ 0 & -\frac{1}{l_2} & \frac{1}{l_2} + \frac{1}{l_3} & -\frac{1}{l_3} \\ 0 & 0 & -\frac{1}{l_3} & \frac{1}{l_3} \end{bmatrix}
 \end{aligned}$$

- Force Term

$$\mathbf{F}_e = \frac{1}{l_e} \int_{x_e}^{x_{e+1}} \left(\frac{x_{e+1} - x}{x - x_e} \right) \cdot 1 dx = \frac{l_e}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\mathbf{F} = \sum_e \mathbf{C}_e^T \mathbf{F}_e = \mathbf{C}_1^T \mathbf{F}_1 + \mathbf{C}_2^T \mathbf{F}_2 + \mathbf{C}_3^T \mathbf{F}_3 = \frac{l_1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{l_2}{2} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{l_3}{2} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} l_1 \\ l_1 + l_2 \\ l_2 + l_3 \\ l_3 \end{pmatrix}$$

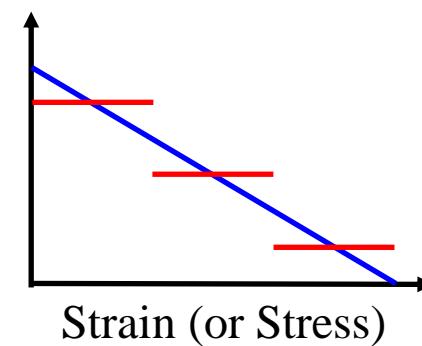
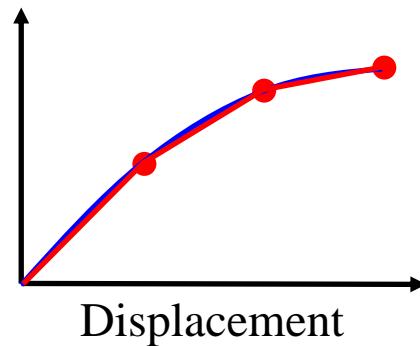
- System Equation

$$\begin{bmatrix} \frac{1}{l_1} & -\frac{1}{l_1} & 0 & 0 \\ -\frac{1}{l_1} & \frac{1}{l_1} + \frac{1}{l_2} & -\frac{1}{l_2} & 0 \\ 0 & -\frac{1}{l_2} & \frac{1}{l_2} + \frac{1}{l_3} & -\frac{1}{l_3} \\ 0 & 0 & -\frac{1}{l_3} & \frac{1}{l_3} \end{bmatrix} \begin{pmatrix} U^1 \\ U^2 \\ U^3 \\ U^4 \end{pmatrix} = \begin{pmatrix} \frac{l_1}{2} \\ \frac{l_1 + l_2}{2} \\ \frac{l_2 + l_3}{2} \\ \frac{l_3}{2} \end{pmatrix}$$

- Case 1 : $l_1 = l_2 = l_3 = \frac{1}{3}$ $u(0) = 0$, $\frac{du(1)}{dx} = 0$

$$\begin{bmatrix} \frac{1}{l_1} & -\frac{1}{l_1} & 0 & 0 \\ -\frac{1}{l_1} & \frac{1}{l_1} + \frac{1}{l_2} & -\frac{1}{l_2} & 0 \\ 0 & -\frac{1}{l_2} & \frac{1}{l_2} + \frac{1}{l_3} & -\frac{1}{l_3} \\ 0 & 0 & -\frac{1}{l_3} & \frac{1}{l_3} \end{bmatrix} \begin{pmatrix} U^1 \\ U^2 \\ U^3 \\ U^4 \end{pmatrix} = \begin{pmatrix} \frac{l_1}{2} \\ \frac{l_1 + l_2}{2} \\ \frac{l_2 + l_3}{2} \\ \frac{l_3}{2} \end{pmatrix} \rightarrow \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{pmatrix} U^2 \\ U^3 \\ U^4 \end{pmatrix} = \frac{1}{9} \begin{pmatrix} 1 \\ 1 \\ 0.5 \end{pmatrix}$$

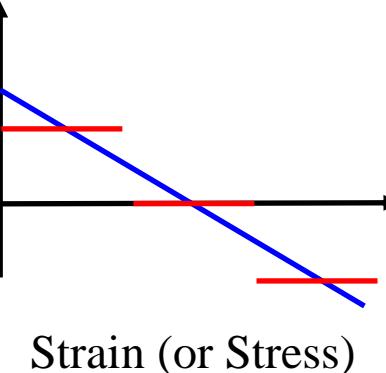
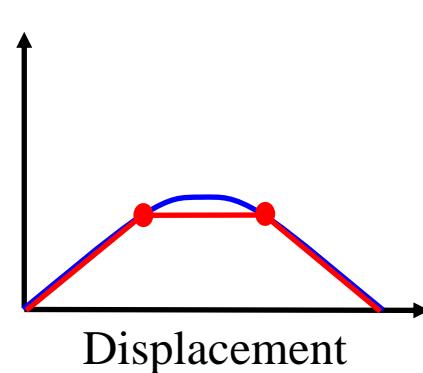
$$U_2 = \frac{5}{18}, U_3 = \frac{8}{18}, U_4 = \frac{9}{18}$$



Case 2 : $l_1 = l_2 = l_3 = \frac{1}{3}$ $u(0) = 0$, $u(1) = 0$

$$\begin{bmatrix} \frac{1}{l_1} & -\frac{1}{l_1} & 0 & 0 \\ -\frac{1}{l_1} & \frac{1}{l_1} + \frac{1}{l_2} & -\frac{1}{l_2} & 0 \\ 0 & -\frac{1}{l_2} & \frac{1}{l_2} + \frac{1}{l_3} & -\frac{1}{l_3} \\ 0 & 0 & -\frac{1}{l_3} & \frac{1}{l_3} \end{bmatrix} \begin{pmatrix} U^1 \\ U^2 \\ U^3 \\ U^4 \end{pmatrix} = \begin{pmatrix} \frac{l_1}{2} \\ \frac{l_1 + l_2}{2} \\ \frac{l_2 + l_3}{2} \\ \frac{l_3}{2} \end{pmatrix} \rightarrow \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{pmatrix} U^3 \\ U^4 \end{pmatrix} = \frac{1}{9} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

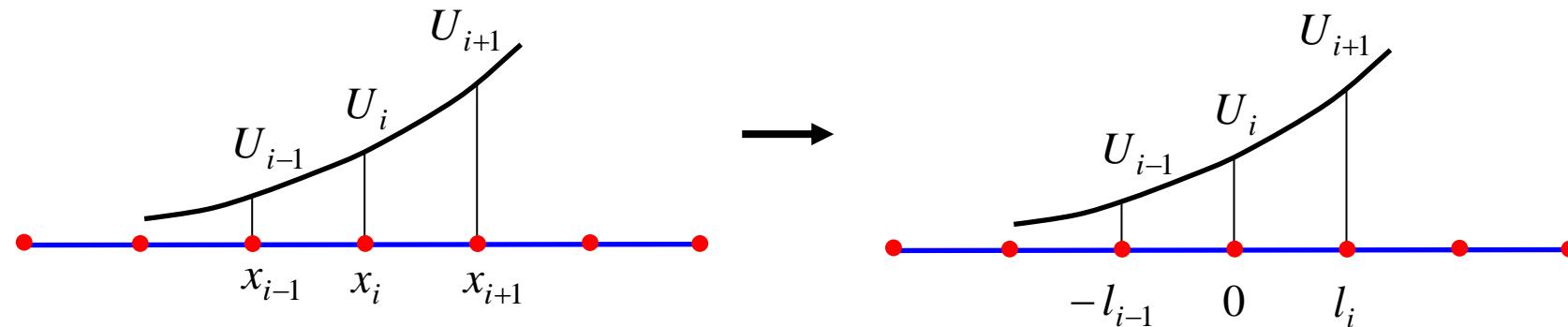
$$U_2 = \frac{1}{9}, U_3 = \frac{1}{9}$$



3.6. Finite Difference Discretization

- Differential Equation**

$$\frac{d^2u}{dx^2} + f = 0 \rightarrow D^2 u^i + f_i = 0 \quad \text{where } D^2 \text{ is a 2nd-order finite difference operator.}$$



- Finite Difference Operator (central Difference)**

- Suppose u is approximated by a 2nd-order parabola, ie, $u \approx ax^2 + bx + c$

$$\left. \begin{array}{l} U_{i-1} = al_{i-1}^2 - bl_{i-1} + c \\ U_i = c \\ U_{i+1} = al_i^2 + bl_i + c \end{array} \right\} \rightarrow a = \frac{1}{l_{i-1} + l_i} \left(\frac{1}{l_{i-1}} U_{i-1} - \left(\frac{1}{l_{i-1}} + \frac{1}{l_i} \right) U_i + \frac{1}{l_i} U_{i+1} \right)$$

$$\left. \frac{d^2u}{dx^2} \right|_{x=x_i} \approx D^2u_i = 2a = \frac{2}{l_{i-1} + l_i} \left(\frac{1}{l_{i-1}} U_{i-1} - \left(\frac{1}{l_{i-1}} + \frac{1}{l_i} \right) U_i + \frac{1}{l_i} U_{i+1} \right)$$

- **Finite Difference Equations for Interior Nodes**

$$-\frac{1}{l_{i-1}} U_{i-1} + \left(\frac{1}{l_{i-1}} + \frac{1}{l_i} \right) U_i - \frac{1}{l_i} U_{i+1} = \frac{l_{i-1} + l_i}{2} f_i$$

- **Finite Difference Equations for Boundary Nodes with Displacement BCs**

In case that a displacement BC is specified at a boundary node, the finite difference equations need to be set up for only interior nodes. The BC can be applied to the finite difference equation for the node adjacent to the boundary nodes.

- Example (Case 2)

$$-\cancel{\frac{1}{l_1} U_1} + \left(\frac{1}{l_1} + \frac{1}{l_2} \right) U_2 - \frac{1}{l_2} U_3 = \frac{l_1 + l_2}{2} f_2$$

$$-\frac{1}{l_2} U_2 + \left(\frac{1}{l_2} + \frac{1}{l_3} \right) U_3 - \cancel{\frac{1}{l_3} U_4} = \frac{l_2 + l_3}{2} f_3$$

- **Finite Difference Equations for Boundary Nodes with Traction BCs**

In case that a traction BC is specified at a boundary nodes, a special treatment for boundary condition such as a ghost node is introduced.



- The finite difference equation at the node $n+1$:

$$-\frac{1}{l_n}U_n + \left(\frac{1}{l_n} + \frac{1}{l_{n+1}}\right)U_{n+1} - \frac{1}{l_{n+1}}U_{n+2} = \frac{l_n + l_{n+1}}{2} f_{n+1}$$

- Approximation of the traction BC by the finite difference operator.

$$\left. \frac{du}{dx} \right|_{x=l} \approx \frac{U_{n+2} - U_n}{l_{n+1} + l_n} = 0 \rightarrow U_{n+2} = U_n$$

- Substitution of the FD traction BC into FD equations for the boundary node.

$$-\left(\frac{1}{l_n} + \frac{1}{l_{n+1}}\right)U_n + \left(\frac{1}{l_n} + \frac{1}{l_{n+1}}\right)U_{n+1} = \frac{l_n + l_{n+1}}{2} f_{n+1}$$

Since the location of the ghost node is arbitrary, $l_{n+1} = l_n$ can be assumed without loss of generality. The final equation for the boundary node becomes

$$-\frac{1}{l_n}U_n + \frac{1}{l_n}U_{n+1} = \frac{l_n}{2}f_{n+1}$$

- Example (Case 1)

$$-\frac{1}{l_1}U_1 + \left(\frac{1}{l_1} + \frac{1}{l_2}\right)U_2 - \frac{1}{l_2}U_3 = \frac{l_1 + l_2}{2}f_2$$

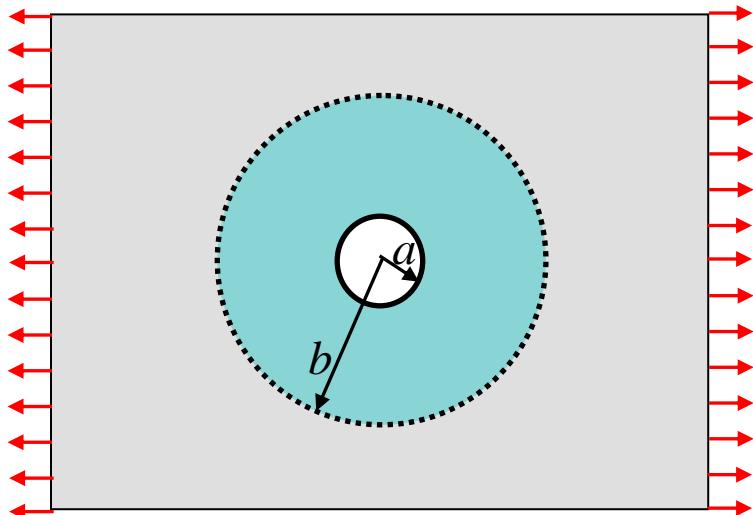
$$-\frac{1}{l_2}U_2 + \left(\frac{1}{l_2} + \frac{1}{l_3}\right)U_3 - \frac{1}{l_3}U_4 = \frac{l_2 + l_3}{2}f_3$$

$$-\frac{1}{l_3}U_3 + \frac{1}{l_3}U_4 = \frac{l_3}{2}f_4$$

Homework 4

Chapter 4

Multidimensional Problems – Elasticity Problems –

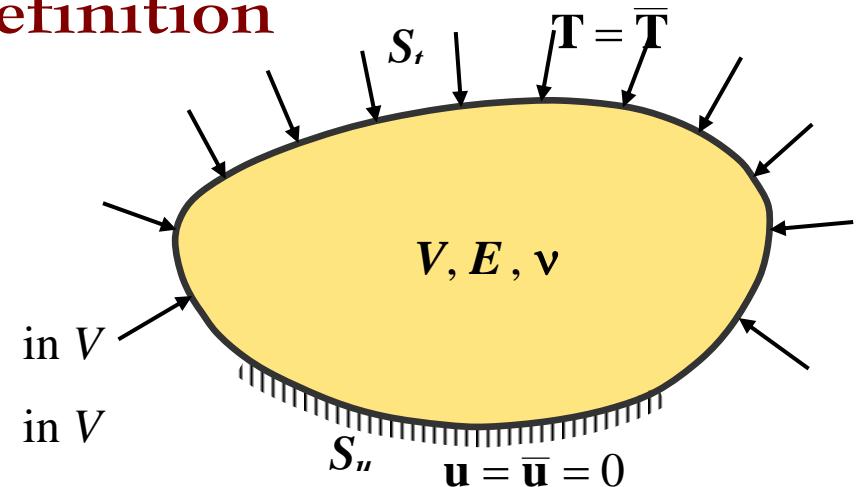


$$\begin{aligned}
 &= \frac{1}{2} \int_V \varepsilon_{ij}^k \sigma_{ij}^k dV - \int_V u_i^k b_i dV - \int_{\Gamma_i} u_i^k \bar{T}_i d\Gamma \\
 &= \frac{1}{2} \int_V \frac{\partial(u_i - u_i^\epsilon)}{\partial x_j} D_{ijkl} \frac{\partial(u_k - u_k^\epsilon)}{\partial x_l} dV - \int_V (u_i - u_i^\epsilon) b_i dV - \int_{\Gamma_i} (u_i - u_i^\epsilon) \bar{T}_i d\Gamma \\
 &= \frac{1}{2} \int_V \frac{\partial u_i}{\partial x_j} D_{ijkl} \frac{\partial u_k}{\partial x_l} dV - \frac{1}{2} \int_V \frac{\partial u_i^\epsilon}{\partial x_j} D_{ijkl} \frac{\partial u_k}{\partial x_l} dV - \frac{1}{2} \int_V \frac{\partial u_i}{\partial x_j} D_{ijkl} \frac{\partial u_k^\epsilon}{\partial x_l} dV \\
 &\quad - \int_V (u_i - u_i^\epsilon) b_i dV - \int_{\Gamma_i} (u_i - u_i^\epsilon) \bar{T}_i d\Gamma \\
 &= \frac{1}{2} \int_V \frac{\partial u_i}{\partial x_j} D_{ijkl} \frac{\partial u_k}{\partial x_l} dV - \int_V u_i b_i dV - \int_{\Gamma_i} u_i \bar{T}_i d\Gamma + \frac{1}{2} \int_V \frac{\partial u_i^\epsilon}{\partial x_j} D_{ijkl} \frac{\partial u_k}{\partial x_l} dV \\
 &\quad - (\int_V \frac{\partial u_i^\epsilon}{\partial x_j} D_{ijkl} \frac{\partial u_k}{\partial x_l} dV - \int_V u_i^\epsilon b_i dV - \int_{\Gamma_i} u_i^\epsilon \bar{T}_i d\Gamma)
 \end{aligned}$$

4.1. Problem Definition

- Governing Equations and Boundary Conditions

Equilibrium Equation : $\nabla \cdot \boldsymbol{\sigma} + \mathbf{b} = 0$



Constitutive Law : $\boldsymbol{\sigma} = \mathbf{D} : \boldsymbol{\varepsilon}$

Strain-Displacement Rel. : $\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T)$ in V

Displacement BC : $\mathbf{u} - \bar{\mathbf{u}} = 0$ on S_u

Traction BC : $\mathbf{T} - \bar{\mathbf{T}} = 0$ on S_t

Cauchy's Relation : $\mathbf{T} = \boldsymbol{\sigma} \cdot \mathbf{n}$ on S

- Strain Energy

$$\Pi_{\text{int}} = \frac{1}{2} \int_V \varepsilon_{ij} \sigma_{ij} dV \leftarrow \varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) = \frac{1}{2} \int_V \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \sigma_{ij} dV = \frac{1}{2} \int_V \frac{\partial u_i}{\partial x_j} \sigma_{ij} dV$$

4.2. Error Minimization

- **Error Estimator**

$$\Pi^R = \int_V (u_i - u_i^h)(\sigma_{ij,j}^h + b_i) dV + \int_S (u_i - u_i^h)(T - T_i^h) dS$$

- **Least Square Error**

$$\begin{aligned}
 \Pi^R &= \frac{1}{2} \int_V (u_i - u_i^h)(\sigma_{ij,j}^h - \sigma_{ij,j}) dV + \frac{1}{2} \int_S (u_i - u_i^h)(T - T_i^h) dS \\
 &= -\frac{1}{2} \int_V (u_{i,j} - u_{i,j}^h)(\sigma_{ij}^h - \sigma_{ij}) dV + \frac{1}{2} \int_S (u_i - u_i^h)(\sigma_{ij}^h - \sigma_{ij}) n_j dS + \frac{1}{2} \int_S (u_i - u_i^h)(T - T_i^h) dS \\
 &= \frac{1}{2} \int_V (u_{i,j} - u_{i,j}^h)(\sigma_{ij}^h - \sigma_{ij}) dV - \frac{1}{2} \int_S (u_i - u_i^h)(T - T_i^h) dS + \frac{1}{2} \int_S (u_i - u_i^h)(T - T_i^h) dS \\
 &= \frac{1}{2} \int_V (u_{i,j} - u_{i,j}^h)(\sigma_{ij}^h - \sigma_{ij}) dV = \frac{1}{2} \int_V (u_{i,j} - u_{i,j}^h) D_{ijkl} (\varepsilon_{kl} - \varepsilon_{kl}^h) dV \\
 &= \frac{1}{2} \int_V (u_{i,j} - u_{i,j}^h) D_{ijkl} (u_{k,l} - u_{k,l}^h) dV = \Pi^{LS}
 \end{aligned}$$

- **Energy Functional – Total Potential Energy**

$$\begin{aligned}
 \Pi^{LS} &= \frac{1}{2} \int_V \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_i^h}{\partial x_j} \right) (\sigma_{ij} - \sigma_{ij}^h) dV = \frac{1}{2} \int_V \frac{\partial u_i}{\partial x_j} \sigma_{ij} dV - \frac{1}{2} \int_V \frac{\partial u_i}{\partial x_j} \sigma_{ij}^h dV - \frac{1}{2} \int_V \frac{\partial u_i^h}{\partial x_j} (\sigma_{ij} - \sigma_{ij}^h) dV \\
 &= \frac{1}{2} \int_V \frac{\partial u_i}{\partial x_j} \sigma_{ij} dV - \frac{1}{2} \int_V \frac{\partial u_i}{\partial x_j} D_{ijkl} \frac{\partial u_k^h}{\partial x_l} dV - \frac{1}{2} \int_V \frac{\partial u_i^h}{\partial x_j} (\sigma_{ij} - \sigma_{ij}^h) dV \\
 &= \frac{1}{2} \int_V \frac{\partial u_i}{\partial x_j} \sigma_{ij} dV - \frac{1}{2} \int_V \frac{\partial u_i^h}{\partial x_j} D_{ijkl} \frac{\partial u_k}{\partial x_l} dV - \frac{1}{2} \int_V \frac{\partial u_i^h}{\partial x_j} (\sigma_{ij} - \sigma_{ij}^h) dV \\
 &= \frac{1}{2} \int_V \frac{\partial u_i}{\partial x_j} \sigma_{ij} dV - \frac{1}{2} \int_V \frac{\partial u_i^h}{\partial x_j} \sigma_{ij} dV - \frac{1}{2} \int_V \frac{\partial u_i^h}{\partial x_j} (\sigma_{ij} - \sigma_{ij}^h) dV \\
 &= \frac{1}{2} \int_V \frac{\partial u_i}{\partial x_j} \sigma_{ij} dV + \frac{1}{2} \int_V \frac{\partial u_i^h}{\partial x_j} \sigma_{ij}^h dV - \int_V \frac{\partial u_i^h}{\partial x_j} \sigma_{ij} dV \\
 &= \frac{1}{2} \int_V \frac{\partial u_i}{\partial x_j} \sigma_{ij} dV + \frac{1}{2} \int_V \varepsilon_{ij}^h \sigma_{ij}^h dV + \int_V u_i^h \frac{\partial \sigma_{ij}}{\partial x_j} dV - \int_S u_i^h \sigma_{ij} n_j dS \\
 &= \frac{1}{2} \int_V \frac{\partial u_i}{\partial x_j} \sigma_{ij} dV + \frac{1}{2} \int_V \varepsilon_{ij}^h \sigma_{ij}^h dV - \int_V u_i^h b_i dV - \int_{S_t} u_i^h \bar{T}_i dS = C + \Pi^{RR}
 \end{aligned}$$

- **Minimization Problems**

$$\text{Min } \Pi^R \Leftrightarrow \text{Min } \Pi^{LS} \Leftrightarrow \text{Min } \Pi^{RR} \text{ w.r.t. } u_i^h \in \mathcal{V}_i^h$$

- Min Π^{RR} : *Rayleigh-Ritz Method* or *Principle of Minimum Potential Energy*

Find $\mathbf{u}^h \in \mathcal{V}^h$ such that minimize Π^{RR}

where $\mathbf{u} \in \mathcal{V} \equiv \{\mathbf{u} \mid \mathbf{u} = 0 \text{ on } S_u\}$, $\left| \int_V \frac{du_i}{dx_j} D_{ijkl} \frac{du_k}{dx_l} dV \right| < \infty \}$

- $\mathbf{v}^h \equiv \mathcal{V}$: The exact solution.
- $\mathbf{v}^h \subset \mathcal{V}$: An approximate solution.

- $\delta\Pi^{RR} = 0$: *Variational Principle* or *Principle of Virtual Work*

$$\begin{aligned} \delta\Pi^{RR} &= \delta \left(\frac{1}{2} \int_V \epsilon_{ij}^h \sigma_{ij}^h dV - \int_V u_i^h b_i dV - \int_{S_t} u_i^h \bar{T}_i dS \right) = \frac{1}{2} \int_V (\delta \epsilon_{ij}^h \sigma_{ij}^h + \epsilon_{ij}^h \delta \sigma_{ij}^h) dV - \int_V \delta u_i^h b_i dV - \int_{S_t} \delta u_i^h \bar{T}_i dS \\ &= \frac{1}{2} \int_V (\delta \epsilon_{ij}^h D_{ijkl} \epsilon_{kl}^h + \epsilon_{ij}^h D_{ijkl} \delta \epsilon_{kl}^h) dV - \int_V \delta u_i^h b_i dV - \int_{S_t} \delta u_i^h \bar{T}_i dS \\ &= \int_V \delta \epsilon_{ij}^h D_{ijkl} \epsilon_{kl}^h dV - \int_V \delta u_i^h b_i dV - \int_{S_t} \delta u_i^h \bar{T}_i dS = \int_V \delta \epsilon_{ij}^h \sigma_{ij}^h dV - \int_V \delta u_i^h b_i dV - \int_{S_t} \delta u_i^h \bar{T}_i dS = 0 \end{aligned}$$

- **Absolute Minimum Property of the Total Potential Energy:** $u_i^h = u_i - u_i^e$

$$\begin{aligned}
 \Pi^h &= \frac{1}{2} \int_V \varepsilon_{ij}^h \sigma_{ij}^h dV - \int_V u_i^h b_i dV - \int_{S_t} u_i^h \bar{T}_i dS \\
 &= \frac{1}{2} \int_V \frac{\partial(u_i - u_i^e)}{\partial x_j} D_{ijkl} \frac{\partial(u_k - u_k^e)}{\partial x_l} dV - \int_V (u_i - u_i^e) b_i dV - \int_{S_t} (u_i - u_i^e) \bar{T}_i dS \\
 &= \frac{1}{2} \int_V \frac{\partial u_i}{\partial x_j} D_{ijkl} \frac{\partial u_k}{\partial x_l} dV - \frac{1}{2} \int_V \frac{\partial u_i^e}{\partial x_j} D_{ijkl} \frac{\partial u_k}{\partial x_l} dV - \frac{1}{2} \int_V \frac{\partial u_i}{\partial x_j} D_{ijkl} \frac{\partial u_k^e}{\partial x_l} dV + \frac{1}{2} \int_V \frac{\partial u_i^e}{\partial x_j} D_{ijkl} \frac{\partial u_k^e}{\partial x_l} dV \\
 &\quad - \int_V (u_i - u_i^e) b_i dV - \int_{S_t} (u_i - u_i^e) \bar{T}_i dS \\
 &= \frac{1}{2} \int_V \frac{\partial u_i}{\partial x_j} D_{ijkl} \frac{\partial u_k}{\partial x_l} dV - \int_V u_i b_i dV - \int_{S_t} u_i \bar{T}_i dS - \left(\int_V \frac{\partial u_i^e}{\partial x_j} D_{ijkl} \frac{\partial u_k}{\partial x_l} dV - \int_V u_i^e b_i dV - \int_{S_t} u_i^e \bar{T}_i dS \right) \\
 &\quad + \frac{1}{2} \int_V \frac{\partial u_i^e}{\partial x_j} D_{ijkl} \frac{\partial u_k^e}{\partial x_l} dV \\
 &= \Pi^E + \frac{1}{2} \int_V \frac{\partial u_i^e}{\partial x_j} D_{ijkl} \frac{\partial u_k^e}{\partial x_l} dV - \left(\int_V \frac{\partial u_i^e}{\partial x_j} \sigma_{ij} dV - \int_V u_i^e b_i dV - \int_{S_t} u_i^e \bar{T}_i dS \right)
 \end{aligned}$$

$$\begin{aligned}
&= \Pi^E + \frac{1}{2} \int_V \frac{\partial u_i^e}{\partial x_j} D_{ijkl} \frac{\partial u_k^e}{\partial x_l} dV - \left(\int_V \frac{\partial u_i^e}{\partial x_j} \sigma_{ij} dV - \int_V u_i^e b_i dV - \int_{S_t} u_i^e \bar{T}_i dS \right) \\
&= \Pi^E + \frac{1}{2} \int_V \frac{\partial u_i^e}{\partial x_j} D_{ijkl} \frac{\partial u_k^e}{\partial x_l} dV - \left(- \int_V u_i^e \sigma_{ij,j} dV + \int_S u_i^e \sigma_{ij} n_j dS - \int_V u_i^e b_i dV - \int_{S_t} u_i^e \bar{T}_i dS \right) \\
&= \Pi^E + \frac{1}{2} \int_V \frac{\partial u_i^e}{\partial x_j} D_{ijkl} \frac{\partial u_k^e}{\partial x_l} dV - \left(- \int_V u_i^e \sigma_{ij,j} dV + \int_{S_t} u_i^e \bar{T}_i dS - \int_V u_i^e b_i dV - \int_{S_t} u_i^e \bar{T}_i dS \right) \\
&= \Pi^E + \frac{1}{2} \int_V \frac{\partial u_i^e}{\partial x_j} D_{ijkl} \frac{\partial u_k^e}{\partial x_l} dV + \left(\int_V u_i^e \sigma_{ij,j} dV + \int_V u_i^e b_i dV \right) \\
&= \Pi^E + \frac{1}{2} \int_V \frac{\partial u_i^e}{\partial x_j} D_{ijkl} \frac{\partial u_k^e}{\partial x_l} dV + \int_V u_i^e (\sigma_{ij,j} + b_i) dV \\
&= \Pi^E + \frac{1}{2} \int_V \frac{\partial u_i^e}{\partial x_j} D_{ijkl} \frac{\partial u_k^e}{\partial x_l} dV
\end{aligned}$$

If D_{ijkl} is positive definite, $\frac{\partial u_i^e}{\partial x_j} D_{ijkl} \frac{\partial u_k^e}{\partial x_l} > 0 \quad \forall \frac{\partial u_i^e}{\partial x_j} \neq 0$ and $\int_V \frac{\partial u_i^e}{\partial x_j} D_{ijkl} \frac{\partial u_k^e}{\partial x_l} dV = 0$ iff $\frac{\partial u_i^e}{\partial x_j} \equiv 0$

$$\Pi^h \geq \Pi^E \quad (\text{The equality sign holds only for } u_i^h = u_i + ??.)$$

4.3. Principle of Virtual Work

- If the following inequality is valid for all real number α , the principle of virtual work holds.

$$\underline{\underline{\Pi^{RR}(u_i + \alpha v_i) \geq \Pi^{RR}(u_i) \quad \forall v_i \in \mathcal{V}}}$$

$$g(\alpha) \equiv \Pi^{RR}(u_i + \alpha v_i) = \frac{1}{2} \int_V \frac{\partial(u_i + \alpha v_i)}{\partial x_j} D_{ijkl} \frac{\partial(u_k + \alpha v_k)}{\partial x_l} dV - \int_V (u_i + \alpha v_i) b_i dV - \int_{S_t} (u_i + \alpha v_i) \bar{T}_i dS$$

$$\begin{aligned} &= \frac{1}{2} \int_V \frac{\partial u_i}{\partial x_j} D_{ijkl} \frac{\partial u_k}{\partial x_l} dV + \alpha \int_V \frac{\partial v_i}{\partial x_j} D_{ijkl} \frac{\partial u_k}{\partial x_l} dV + \frac{1}{2} \alpha^2 \int_V \frac{\partial v_i}{\partial x_j} D_{ijkl} \frac{\partial v_i}{\partial x_l} dV \\ &\quad - \int_V (u_i + \alpha v_i) b_i dV - \int_{S_t} (u_i + \alpha v_i) \bar{T}_i dS \end{aligned}$$

$$g'(\alpha) = \int_V \frac{\partial v_i}{\partial x_j} D_{ijkl} \frac{\partial u_k}{\partial x_l} dV - \int_V v_i b_i dV - \int_{S_t} v_i \bar{T}_i dS + \alpha \int_V \frac{\partial v_i}{\partial x_j} D_{ijkl} \frac{\partial v_i}{\partial x_l} dV$$

$$g'(0) = \int_V \frac{\partial v_i}{\partial x_j} D_{ijkl} \frac{\partial u_k}{\partial x_l} dV - \int_V v_i b_i dV - \int_{S_t} v_i \bar{T}_i dS = 0 \quad \forall v_i \in \mathcal{V} \text{ (for all admissible } v_i \text{)}$$

If the principle of virtual work holds, then the principle of minimum potential energy holds because the boxed equation of the total potential energy vanishes identically. The approximate version of the principle of virtual work is

$$\int_V \frac{\partial v_i^h}{\partial x_j} \sigma_{ij}^h dV - \int_V v_i^h b_i dV - \int_{S_t} v_i^h \bar{T}_i dS = 0 \quad \text{for all admissible } v_i^h$$

- **Equivalence to the PDE**

- Exact form

$$\int_V v_i (\sigma_{ij,j} + b_i) dV - \int_{\Gamma_t} v_i (T_i - \bar{T}_i) dS = 0 \quad \forall v_i \in \mathcal{V} \rightarrow \underline{\underline{\sigma_{ij,j} + b_i = 0, T_i = \bar{T}_i}}$$

- Approximate form

$$\int_V v_i^h (\sigma_{ij,j}^h + b_i) dV - \int_{\Gamma_t} v_i^h (T_i^h - \bar{T}_i) dS = 0 \quad \forall v_i^h \in \mathcal{V}^h \rightarrow \underline{\underline{\sigma_{ij,j} + b_i \neq 0, T_i \neq \bar{T}_i}}$$

since $\int_V v_i (\sigma_{ij,j}^h + b_i) dV - \int_{\Gamma_t} v_i (T_i^h - \bar{T}_i) dS \neq 0 \quad \forall v_i \in \mathcal{V}$.

- **Equivalence to the Weighted Residual Method:** $v_i^h = \delta a_{ik} g_k$, $u_i^h = a_{ik} g_k$

$$\int_V \delta a_{ik} g_k (\sigma_{ij,j}^h + b_i) dV = \int_{S_t} \delta a_{ik} g_k (T_i^h - \bar{T}_i) dS \text{ for possible } \delta a_{ik} \rightarrow$$

$$\int_V g_k (\sigma_{ij,j}^h + b_i) dV = \int_{S_t} g_k (T_i^h - \bar{T}_i) dS \quad \text{for all } k$$

- **Uniqueness of solution**

If two solutions satisfy the principle of virtual work, then

$$\int_V \frac{\partial v_i}{\partial x_j} \sigma_{ij}^1 dV - \int_V v_i b_i dV - \int_{S_t} v_i \bar{T}_i dS = 0 \quad \text{for all admissible } v_i$$

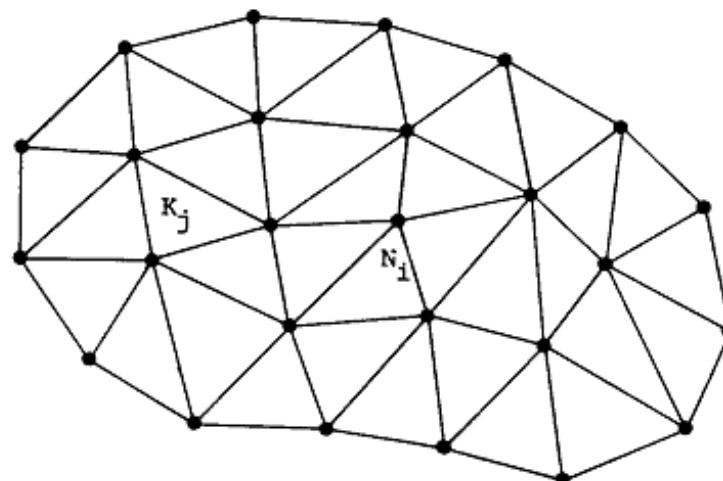
$$\int_V \frac{\partial v_i}{\partial x_j} \sigma_{ij}^2 dV - \int_V v_i b_i dV - \int_{S_t} v_i \bar{T}_i dS = 0 \quad \text{for all admissible } v_i$$

By subtracting two equations, $\int_V \frac{\partial v_i}{\partial x_j} (\sigma_{ij}^1 - \sigma_{ij}^2) dV = 0$ for all admissible v_i $\rightarrow \sigma_{ij}^1 - \sigma_{ij}^2 = 0$

$$D_{ijkl} \left(\frac{\partial u_k^1}{\partial x_l} - \frac{\partial u_k^2}{\partial x_l} \right) = 0 \rightarrow \frac{\partial u_k^1}{\partial x_l} - \frac{\partial u_k^2}{\partial x_l} = 0$$

Chapter 5

Discretization



Rayleigh-Ritz Type Discretization

- **Approximation**

$$u_i^h = c_{i1}g_1 + c_{i2}g_2 + \cdots + c_{in}g_n = \sum_{p=1}^n c_{ip}g_p, \quad \frac{\partial u_i^h}{\partial x_j} = \sum_{p=1}^n c_{ip} \frac{\partial g_p}{\partial x_j} = c_{ip} \frac{\partial g_p}{\partial x_j}$$

- **Principle of Minimum Potential Energy**

$$\text{Min } \Pi^h = \frac{1}{2} \int_V c_{ip} \frac{\partial g_p}{\partial x_j} D_{ijkl} c_{kq} \frac{\partial g_q}{\partial x_l} dV - \int_V c_{ip} g_p b_i dV - \int_{S_t} c_{ip} g_p \bar{T}_i dS \quad \text{or} \quad \frac{\partial \Pi^h}{\partial c_{mr}} = 0 \quad \text{for all } m, r$$

$$\frac{\partial \Pi^h}{\partial c_{mr}} = \frac{1}{2} \int_V \delta_{mi} \delta_{rp} \frac{\partial g_p}{\partial x_j} D_{ijkl} c_{kq} \frac{\partial g_q}{\partial x_l} dV + \frac{1}{2} \int_V c_{ip} \frac{\partial g_p}{\partial x_j} D_{ijkl} \delta_{mk} \delta_{rq} \frac{\partial g_q}{\partial x_l} dV - \int_V \delta_{mi} g_r b_i dV - \int_{S_t} \delta_{mi} g_r \bar{T}_i dS$$

$$= \frac{1}{2} \int_V \frac{\partial g_r}{\partial x_j} D_{mkl} c_{kq} \frac{\partial g_q}{\partial x_l} dV + \frac{1}{2} \int_V c_{ip} \frac{\partial g_p}{\partial x_j} D_{ijml} \frac{\partial g_r}{\partial x_l} dV - \int_V g_r b_m dV - \int_{S_t} g_r \bar{T}_m dS$$

$$= \int_V \frac{\partial g_r}{\partial x_j} D_{mjk} c_{kp} \frac{\partial g_p}{\partial x_l} dV - \int_V g_r b_m dV - \int_{S_t} g_r \bar{T}_m dS$$

$$= \int_V \frac{\partial g_r}{\partial x_j} D_{mjk} \frac{\partial g_p}{\partial x_l} dV c_{kp} - \int_V g_r b_m dV - \int_{S_t} g_r \bar{T}_m dS = K_{rmkp} c_{kp} - f_{rm} = 0 \quad \text{for all } r \text{ and } m$$

- Principle of Virtual Work

$$\int_V \delta \varepsilon_{ij}^h \sigma_{ij}^h dV - \int_V \delta u_i^h b_i^h dV - \int_V \delta u_i^h \bar{T}_i dV = 0$$

$$\begin{aligned}\delta \Pi^h &= \int_V \delta c_{ip} \frac{\partial g_p}{\partial x_j} D_{ijkl} c_{kq} \frac{\partial g_q}{\partial x_l} dV - \int_V \delta c_{ip} g_p b_i dV - \int_{S_t} \delta c_{ip} g_p \bar{T}_i dS \\ &= \delta c_{ip} \left(\int_V \frac{\partial g_p}{\partial x_j} D_{ijkl} \frac{\partial g_q}{\partial x_l} dV c_{kq} - \int_V g_p b_i dV - \int_{S_t} g_p \bar{T}_i dS \right) \\ &= \delta c_{ip} \left(\int_V \frac{\partial g_p}{\partial x_j} D_{ijkl} \frac{\partial g_q}{\partial x_l} dV c_{kq} - \int_V g_p b_i dV - \int_{S_t} g_p \bar{T}_i dS \right) = 0 \quad \text{for all } \delta c_{ip}\end{aligned}$$

$$K_{pikq} c_{kq} - f_{pi} = 0 \quad \text{for all } p \text{ and } i$$

- **Matrix Form – Virtual Work Expression**

$$\int_V \delta \varepsilon_{ij}^h \sigma_{ij}^h dV =$$

$$\int_V (\delta \varepsilon_{11}^h \sigma_{11}^h + \delta \varepsilon_{22}^h \sigma_{22}^h + \delta \varepsilon_{33}^h \sigma_{33}^h + \delta \varepsilon_{12}^h \sigma_{12}^h + \delta \varepsilon_{21}^h \sigma_{21}^h + \delta \varepsilon_{13}^h \sigma_{13}^h + \delta \varepsilon_{31}^h \sigma_{31}^h + \delta \varepsilon_{23}^h \sigma_{23}^h + \delta \varepsilon_{32}^h \sigma_{32}^h) dV =$$

$$\int_V (\delta \varepsilon_{11}^h \sigma_{11}^h + \delta \varepsilon_{22}^h \sigma_{22}^h + \delta \varepsilon_{33}^h \sigma_{33}^h + 2\delta \varepsilon_{12}^h \sigma_{12}^h + 2\delta \varepsilon_{13}^h \sigma_{13}^h + 2\delta \varepsilon_{23}^h \sigma_{23}^h) dV =$$

$$\int_V (\delta \varepsilon_{11}^h \sigma_{11}^h + \delta \varepsilon_{22}^h \sigma_{22}^h + \delta \varepsilon_{33}^h \sigma_{33}^h + \delta \gamma_{12}^h \sigma_{12}^h + \delta \gamma_{13}^h \sigma_{13}^h + \delta \gamma_{23}^h \sigma_{23}^h) dV = \int_V \delta \boldsymbol{\varepsilon}^h \cdot \boldsymbol{\sigma}^h dV$$

$$\int_V \delta \boldsymbol{\varepsilon}^h \cdot \boldsymbol{\sigma}^h dV = \int_V \delta \mathbf{u}^h \cdot \mathbf{b}^h dV + \int_{S_t} \delta \mathbf{u}^h \cdot \bar{\mathbf{T}} dS$$

- **Displacement**

$$\mathbf{u}^h = \begin{pmatrix} u_1^h \\ u_2^h \\ u_3^h \end{pmatrix} = \begin{bmatrix} g_1 & 0 & 0 & g_n & 0 & 0 \\ 0 & g_1 & 0 & \cdots & 0 & g_n \\ 0 & 0 & g_1 & 0 & 0 & g_n \end{bmatrix} \begin{pmatrix} c_{11} \\ c_{21} \\ c_{31} \\ \vdots \\ c_{1n} \\ c_{2n} \\ c_{3n} \end{pmatrix} = \mathbf{Nc}$$

- Virtual Strain

$$(\delta \boldsymbol{\varepsilon}^h) = \begin{pmatrix} \delta \varepsilon_{11}^h \\ \delta \varepsilon_{22}^h \\ \delta \varepsilon_{33}^h \\ \delta \gamma_{12}^h \\ \delta \gamma_{13}^h \\ \delta \gamma_{23}^h \end{pmatrix} = \begin{pmatrix} \frac{\partial \delta u_1^h}{\partial x_1} \\ \frac{\partial \delta u_2^h}{\partial x_2} \\ \frac{\partial \delta u_3^h}{\partial x_3} \\ \frac{\partial \delta u_1^h}{\partial x_2} + \frac{\partial \delta u_2^h}{\partial x_1} \\ \frac{\partial \delta u_1^h}{\partial x_3} + \frac{\partial \delta u_3^h}{\partial x_1} \\ \frac{\partial \delta u_2^h}{\partial x_3} + \frac{\partial \delta u_3^h}{\partial x_2} \end{pmatrix} = \begin{pmatrix} \delta c_{1i} \frac{\partial g_i}{\partial x_1} \\ \delta c_{2i} \frac{\partial g_i}{\partial x_2} \\ \delta c_{3i} \frac{\partial g_i}{\partial x_3} \\ \delta c_{1i} \frac{\partial g_i}{\partial x_2} + \delta c_{2i} \frac{\partial g_i}{\partial x_1} \\ \delta c_{1i} \frac{\partial g_i}{\partial x_3} + \delta c_{3i} \frac{\partial g_i}{\partial x_1} \\ \delta c_{2i} \frac{\partial g_i}{\partial x_3} + \delta c_{3i} \frac{\partial g_i}{\partial x_2} \end{pmatrix}$$

- Virtual Strain – Matrix Form

$$(\delta \varepsilon^h) = \begin{pmatrix} \delta \varepsilon_{11}^h \\ \delta \varepsilon_{22}^h \\ \delta \varepsilon_{33}^h \\ \delta \gamma_{12}^h \\ \delta \gamma_{13}^h \\ \delta \gamma_{23}^h \end{pmatrix} = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & 0 & 0 & \frac{\partial g_n}{\partial x_1} & 0 & 0 \\ 0 & \frac{\partial g_1}{\partial x_2} & 0 & 0 & \frac{\partial g_n}{\partial x_2} & 0 \\ 0 & 0 & \frac{\partial g_1}{\partial x_3} & \dots & 0 & \frac{\partial g_n}{\partial x_3} \\ \frac{\partial g_1}{\partial x_2} & \frac{\partial g_1}{\partial x_1} & 0 & \frac{\partial g_n}{\partial x_2} & \frac{\partial g_n}{\partial x_1} & 0 \\ \frac{\partial g_1}{\partial x_3} & 0 & \frac{\partial g_1}{\partial x_1} & \frac{\partial g_n}{\partial x_3} & 0 & \frac{\partial g_n}{\partial x_1} \\ 0 & \frac{\partial g_1}{\partial x_3} & \frac{\partial g_1}{\partial x_2} & 0 & \frac{\partial g_n}{\partial x_3} & \frac{\partial g_n}{\partial x_2} \end{bmatrix} \begin{pmatrix} \delta c_{11} \\ \delta c_{21} \\ \delta c_{31} \\ \vdots \\ \delta c_{1n} \\ \delta c_{2n} \\ \delta c_{3n} \end{pmatrix} = \mathbf{B} \delta \mathbf{c}$$

- Stress-strain (displacement) Relation

$$(\boldsymbol{\sigma}^h) = \begin{pmatrix} \sigma_{11}^h \\ \sigma_{22}^h \\ \sigma_{33}^h \\ \sigma_{12}^h \\ \sigma_{13}^h \\ \sigma_{23}^h \end{pmatrix} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \times \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 0 & 0 & 0 \\ \frac{\nu}{1-\nu} & 1 & \frac{\nu}{1-\nu} & 0 & 0 & 0 \\ \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} \end{bmatrix} \begin{pmatrix} \epsilon_{11}^h \\ \epsilon_{22}^h \\ \epsilon_{33}^h \\ \gamma_{12}^h \\ \gamma_{13}^h \\ \gamma_{23}^h \end{pmatrix}$$

$$= \mathbf{D}\boldsymbol{\epsilon}^h = \mathbf{DBc}$$

- Final System Equation

$$\int_V \delta \varepsilon_{ij}^h \sigma_{ij}^h dV = \int_V \delta \boldsymbol{\varepsilon}^h{}^T \boldsymbol{\sigma}^h dV = \delta \mathbf{c}^T \int_V \mathbf{B}^T \mathbf{D} \mathbf{B} dV \mathbf{c}$$

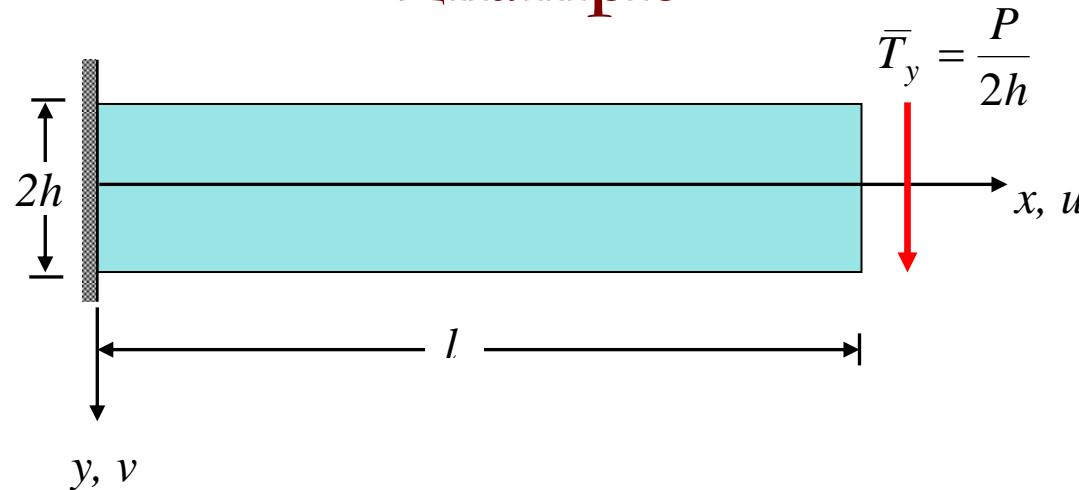
$$\int_V \delta u_i^h b_i dV = \int_V \delta \mathbf{u}^h{}^T \mathbf{b} dV = \delta \mathbf{c}^T \int_V \mathbf{N}^T \mathbf{b} dV$$

$$\int_{\Gamma_t} \delta u_i^h \bar{T}_i d\Gamma = \int_{\Gamma_t} \delta \mathbf{u}^h{}^T \bar{\mathbf{T}} d\Gamma = \delta \mathbf{c}^T \int_{\Gamma_t} \mathbf{N}^T \bar{\mathbf{T}} d\Gamma$$

$$\delta \mathbf{c}^T \left(\int_V \mathbf{B}^T \mathbf{D} \mathbf{B} dV \mathbf{c} - \int_V \mathbf{N}^T \mathbf{b} dV - \int_{\Gamma_t} \mathbf{N}^T \bar{\mathbf{T}} d\Gamma \right) = 0 \text{ for all admissible } \delta \mathbf{c} \text{ or}$$

$$\mathbf{K} \mathbf{c} = \mathbf{f}$$

Example



By the elementary beam solution, the displacement field of the structure is assumed as

$$u = a\left(\frac{x^2}{2} - lx\right)y, \quad v = b\left(\frac{x^3}{6} - \frac{x^2}{2}l\right)$$

$$\mathbf{N} = \begin{bmatrix} \left(\frac{x^2}{2} - lx\right)y & 0 \\ 0 & \frac{x^3}{6} - \frac{x^2}{2}l \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} (x-l)y & 0 \\ 0 & 0 \\ \frac{x^2}{2} - lx & \frac{x^2}{2} - lx \end{bmatrix}, \quad \mathbf{D} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

$$\begin{aligned}
 \mathbf{K} &= \frac{E}{1-\nu^2} \int_0^l \int_{-h}^h \int_0^1 \begin{bmatrix} (x-l)y & 0 & \frac{x^2}{2} - lx \\ 0 & 0 & \frac{x^2}{2} - lx \end{bmatrix} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} (x-l)y & 0 \\ 0 & 0 \\ \frac{x^2}{2} - lx & \frac{x^2}{2} - lx \end{bmatrix} dz dy dx \\
 &= \frac{E}{1-\nu^2} \int_0^l \int_{-h}^h \int_0^1 \begin{bmatrix} (x-l)^2 y^2 + \frac{1-\nu}{2} (\frac{x^2}{2} - lx)^2 & \frac{1-\nu}{2} (\frac{x^2}{2} - lx)^2 \\ \frac{1-\nu}{2} (\frac{x^2}{2} - lx)^2 & \frac{1-\nu}{2} (\frac{x^2}{2} - lx)^2 \end{bmatrix} dz dy dx \\
 &= \frac{E}{1-\nu^2} \begin{bmatrix} 2 \frac{l^3}{3} \frac{h^3}{3} + \frac{1-\nu}{2} \frac{2}{15} l^5 (2h) & \frac{1-\nu}{2} \frac{2}{15} l^5 (2h) \\ \frac{1-\nu}{2} \frac{2}{15} l^5 (2h) & \frac{1-\nu}{2} \frac{2}{15} l^5 (2h) \end{bmatrix} \\
 \mathbf{f} &= \int_{-h}^h \int_0^1 \begin{bmatrix} -\frac{l^2}{2} y & 0 \\ 0 & -\frac{l^3}{3} \end{bmatrix} \begin{pmatrix} 0 \\ P \\ \hline 2h \end{pmatrix} dx dy = -\frac{l^3}{3} \begin{pmatrix} 0 \\ P \end{pmatrix}
 \end{aligned}$$

$$\frac{E}{1-\nu^2} \begin{bmatrix} 2\frac{l^3}{3}\frac{h^3}{3} + \frac{1-\nu}{2}\frac{2}{15}l^5(2h) & \frac{1-\nu}{2}\frac{2}{15}l^5(2h) \\ \frac{1-\nu}{2}\frac{2}{15}l^5(2h) & \frac{1-\nu}{2}\frac{2}{15}l^5(2h) \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = -\frac{l^3}{3} \begin{pmatrix} 0 \\ P \end{pmatrix}$$

$$a = \frac{3}{2} \frac{1-\nu^2}{Eh^3} P = \frac{1-\nu^2}{EI} P , \quad b = -(1 + \frac{5}{3} \frac{1}{1-\nu} (\frac{h}{l})^2) \frac{1-\nu^2}{EI} P$$

$$u = \frac{1-\nu^2}{EI} P \left(\frac{x^2}{2} - lx \right) y , \quad v = -(1 + \frac{5}{3} \frac{1}{1-\nu} (\frac{h}{l})^2) \frac{1-\nu^2}{EI} P \left(\frac{x^3}{6} - \frac{x^2}{2} l \right)$$

$$\sigma_{xx} = \frac{P(x-l)}{I} y = \frac{M}{I} y$$

$$\sigma_{yy} = \nu \frac{M}{I} y$$

$$\tau_{xy} = -\frac{5}{6} \left(\frac{h}{l}\right)^2 \frac{1}{I} \left(\frac{x^2}{2} - xl\right) P$$

Homework 5

Finite Element Discretization

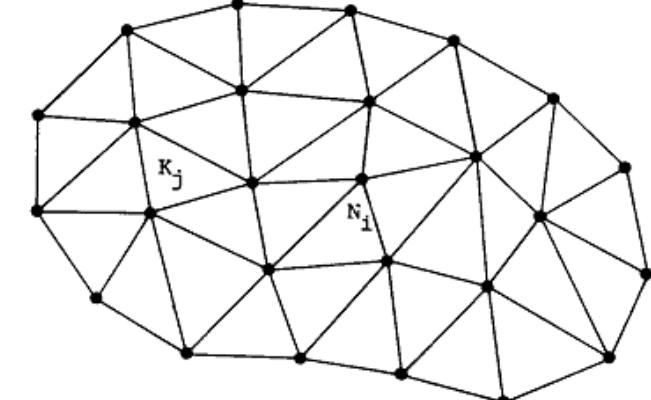
- **Domain Discretization**

$$V = V^1 \cup V^2 \cup \dots \cup V^n, \quad \Gamma = \Gamma^1 \cup \Gamma^2 \cup \dots \cup \Gamma^m$$

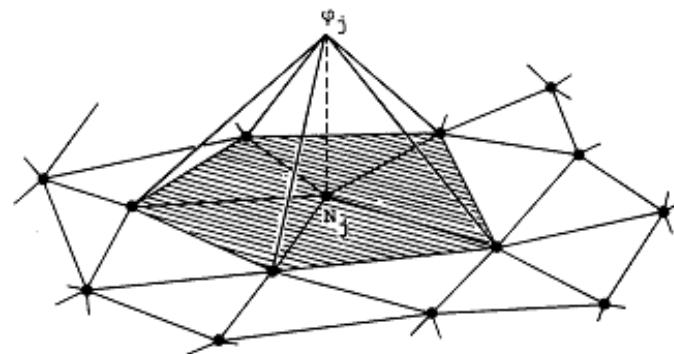
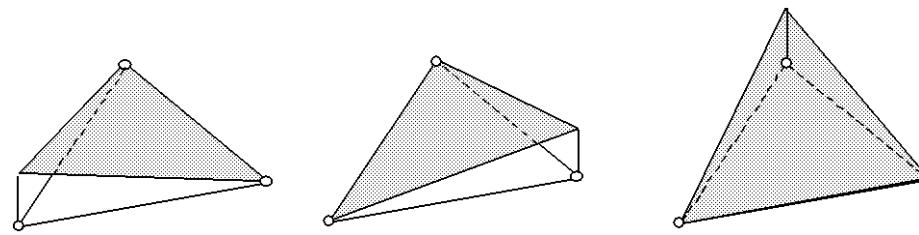
$$\int_V \delta \boldsymbol{\varepsilon}^h{}^T \boldsymbol{\sigma}^h dV = \int_V \delta \mathbf{u}^h{}^T \mathbf{b} dV + \int_{\Gamma_t} \delta \mathbf{u}^h{}^T \bar{\mathbf{T}} d\Gamma \rightarrow \sum_e \int_{V^e} \delta \boldsymbol{\varepsilon}^h{}^T \boldsymbol{\sigma}^h dV = \sum_e \int_{V^e} \delta \mathbf{u}^h{}^T \mathbf{b} dV + \sum_e \int_{S_t^e} \delta \mathbf{u}^h{}^T \mathbf{b} dS$$

- **The displacement field in an element**

$$\mathbf{u}^e = \begin{pmatrix} u_1^e \\ u_2^e \\ u_3^e \end{pmatrix} = \begin{bmatrix} N_1 & 0 & 0 & N_n & 0 & 0 \\ 0 & N_1 & 0 & \dots & 0 & N_n \\ 0 & 0 & N_1 & 0 & 0 & N_n \end{bmatrix} \begin{pmatrix} u_{11} \\ u_{21} \\ u_{31} \\ \vdots \\ u_{1n} \\ u_{2n} \\ u_{3n} \end{pmatrix}^e = \mathbf{N}^e \mathbf{U}^e$$



where $N_i(\mathbf{X}_j) = \delta_{ij}$.



- The virtual strain field in an element

$$\delta \boldsymbol{\varepsilon}^e = \begin{pmatrix} \delta \varepsilon_{11}^e \\ \delta \varepsilon_{22}^e \\ \delta \varepsilon_{33}^e \\ \delta \gamma_{12}^e \\ \delta \gamma_{13}^e \\ \delta \gamma_{23}^e \end{pmatrix} = \begin{pmatrix} \frac{\partial \delta u_1^e}{\partial x_1} \\ \frac{\partial \delta u_2^e}{\partial x_2} \\ \frac{\partial \delta u_3^e}{\partial x_3} \\ \frac{\partial \delta u_1^e}{\partial x_2} + \frac{\partial \delta u_2^e}{\partial x_1} \\ \frac{\partial \delta u_1^e}{\partial x_3} + \frac{\partial \delta u_3^e}{\partial x_1} \\ \frac{\partial \delta u_2^e}{\partial x_3} + \frac{\partial \delta u_3^e}{\partial x_2} \end{pmatrix} = \begin{pmatrix} \delta u_{1i} \frac{\partial N_i}{\partial x_1} \\ \delta u_{2i} \frac{\partial N_i}{\partial x_2} \\ \delta u_{3i} \frac{\partial N_i}{\partial x_3} \\ \delta u_{1i} \frac{\partial N_i}{\partial x_2} + \delta u_{2i} \frac{\partial N_i}{\partial x_1} \\ \delta u_{1i} \frac{\partial N_i}{\partial x_3} + \delta u_{3i} \frac{\partial N_i}{\partial x_1} \\ \delta u_{2i} \frac{\partial N_i}{\partial x_3} + \delta u_{3i} \frac{\partial N_i}{\partial x_2} \end{pmatrix}$$

- The virtual strain field in an element – Matrix Form

$$\delta \boldsymbol{\varepsilon}^e = \begin{pmatrix} \delta \varepsilon_{11}^e \\ \delta \varepsilon_{22}^e \\ \delta \varepsilon_{33}^e \\ \delta \gamma_{12}^e \\ \delta \gamma_{13}^e \\ \delta \gamma_{23}^e \end{pmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial x_1} & 0 & 0 & \frac{\partial N_n}{\partial x_1} & 0 & 0 \\ 0 & \frac{\partial N_1}{\partial x_2} & 0 & 0 & \frac{\partial N_n}{\partial x_2} & 0 \\ 0 & 0 & \frac{\partial N_1}{\partial x_3} & \dots & 0 & \frac{\partial N_n}{\partial x_3} \\ \frac{\partial N_1}{\partial x_2} & \frac{\partial N_1}{\partial x_1} & 0 & \frac{\partial N_n}{\partial x_2} & \frac{\partial N_n}{\partial x_1} & 0 \\ \frac{\partial N_1}{\partial x_3} & 0 & \frac{\partial N_1}{\partial x_1} & \frac{\partial N_n}{\partial x_3} & 0 & \frac{\partial N_n}{\partial x_1} \\ 0 & \frac{\partial N_1}{\partial x_3} & \frac{\partial N_1}{\partial x_2} & 0 & \frac{\partial N_n}{\partial x_3} & \frac{\partial N_n}{\partial x_2} \end{bmatrix} \begin{pmatrix} \delta u_{11} \\ \delta u_{21} \\ \delta u_{31} \\ \vdots \\ \delta u_{1n} \\ \delta u_{2n} \\ \delta u_{3n} \end{pmatrix}^e = \mathbf{B} \delta \mathbf{U}^e$$

- The stress field in an element

$$\boldsymbol{\sigma}^e = \begin{pmatrix} \sigma_{11}^e \\ \sigma_{22}^e \\ \sigma_{33}^e \\ \sigma_{12}^e \\ \sigma_{13}^e \\ \sigma_{23}^e \end{pmatrix} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 0 & 0 & 0 \\ \frac{\nu}{1-\nu} & 1 & \frac{\nu}{1-\nu} & 0 & 0 & 0 \\ \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} \end{bmatrix} \begin{pmatrix} \varepsilon_{11}^e \\ \varepsilon_{22}^e \\ \varepsilon_{33}^e \\ \gamma_{12}^e \\ \gamma_{13}^e \\ \gamma_{23}^e \end{pmatrix}$$

$$= \mathbf{D}\boldsymbol{\varepsilon}^e = \mathbf{DBU}^e$$

- Stiffness equation

$$\sum_e \delta \mathbf{U}^{eT} \int_{V^e} \mathbf{B}^T \mathbf{DB} dV \mathbf{U}^e - \sum_e \delta \mathbf{U}^{eT} \int_{V^e} \mathbf{N}^T \mathbf{b} dV - \sum_e \delta \mathbf{U}^{eT} \int_{S_t^e} \mathbf{N}^T \bar{\mathbf{T}} dS = 0 \rightarrow$$

$$\sum_e \delta \mathbf{U}^{eT} \mathbf{K}^e \mathbf{U}^e - \sum_e \delta \mathbf{U}^{eT} \mathbf{f}^e = 0$$

- **Compatibility conditions**

$$\mathbf{U}^e = \mathbf{T}^e \mathbf{U} \quad , \quad \delta \mathbf{U}^e = \mathbf{T}^e \delta \mathbf{U}$$

$$\delta \mathbf{U}^T \sum_e \mathbf{T}^{eT} \mathbf{K}^e \mathbf{T}^e \mathbf{U} - \delta \mathbf{U}^T \sum_e \mathbf{T}^{eT} \mathbf{f}^e = 0 \text{ for all admissible } \delta \mathbf{U}$$

$$\mathbf{KU} - \mathbf{f} = \mathbf{0}$$

Finite Element Procedure

1. Governing equations in the domain, boundary conditions on the boundary.
2. Derive weak form of the G.E. and B.C. by the variational principle or equivalent.
3. Discretize the given domain and boundary with finite elements.

$$V = V^1 \cup V^2 \cup \dots \cup V^n , \quad S = S^1 \cup S^2 \cup \dots \cup S^n$$

4. Assume the displacement field by shape functions and nodal values within an element.

$$\mathbf{u}^e = \mathbf{N}^e \mathbf{U}^e$$

5. Calculate the element stiffness matrix and assemble it according to the computability.

$$\mathbf{K}^e = \int_{V^e} \mathbf{B}^T \mathbf{D} \mathbf{B} dV , \quad \mathbf{K} = \sum_e \mathbf{K}^e$$

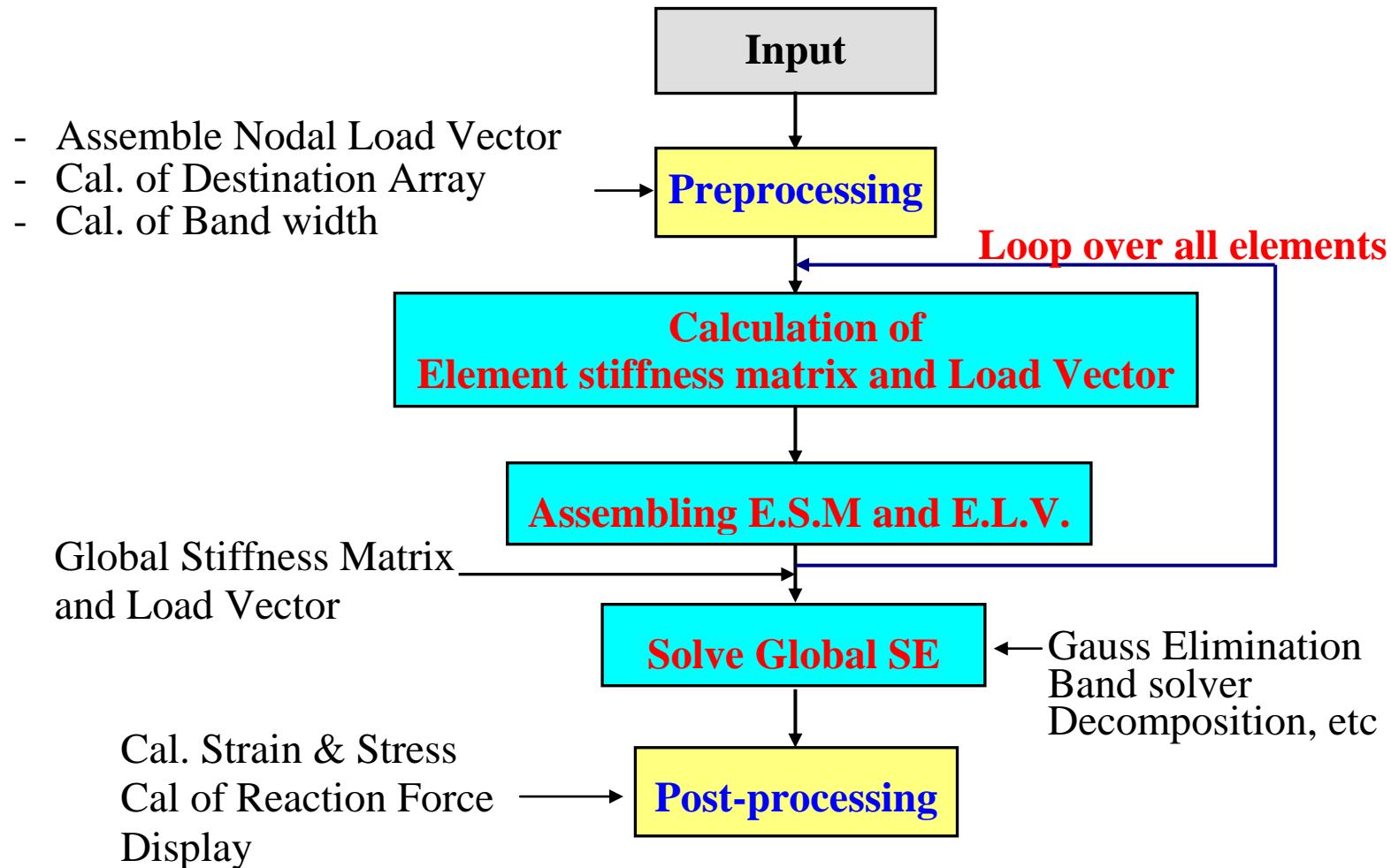
6. Calculate the equivalent nodal force and assemble it according to the computability.

$$\mathbf{f}^e = \int_{V^e} \mathbf{N}^T \mathbf{b} dV + \int_{S_t^e} \mathbf{N}^T \bar{\mathbf{T}} dS , \quad \mathbf{f} = \sum_e \mathbf{f}^e$$

7. Apply the displacement boundary conditions and solve the stiffness equation.
8. Calculate strain, stress and reaction force.

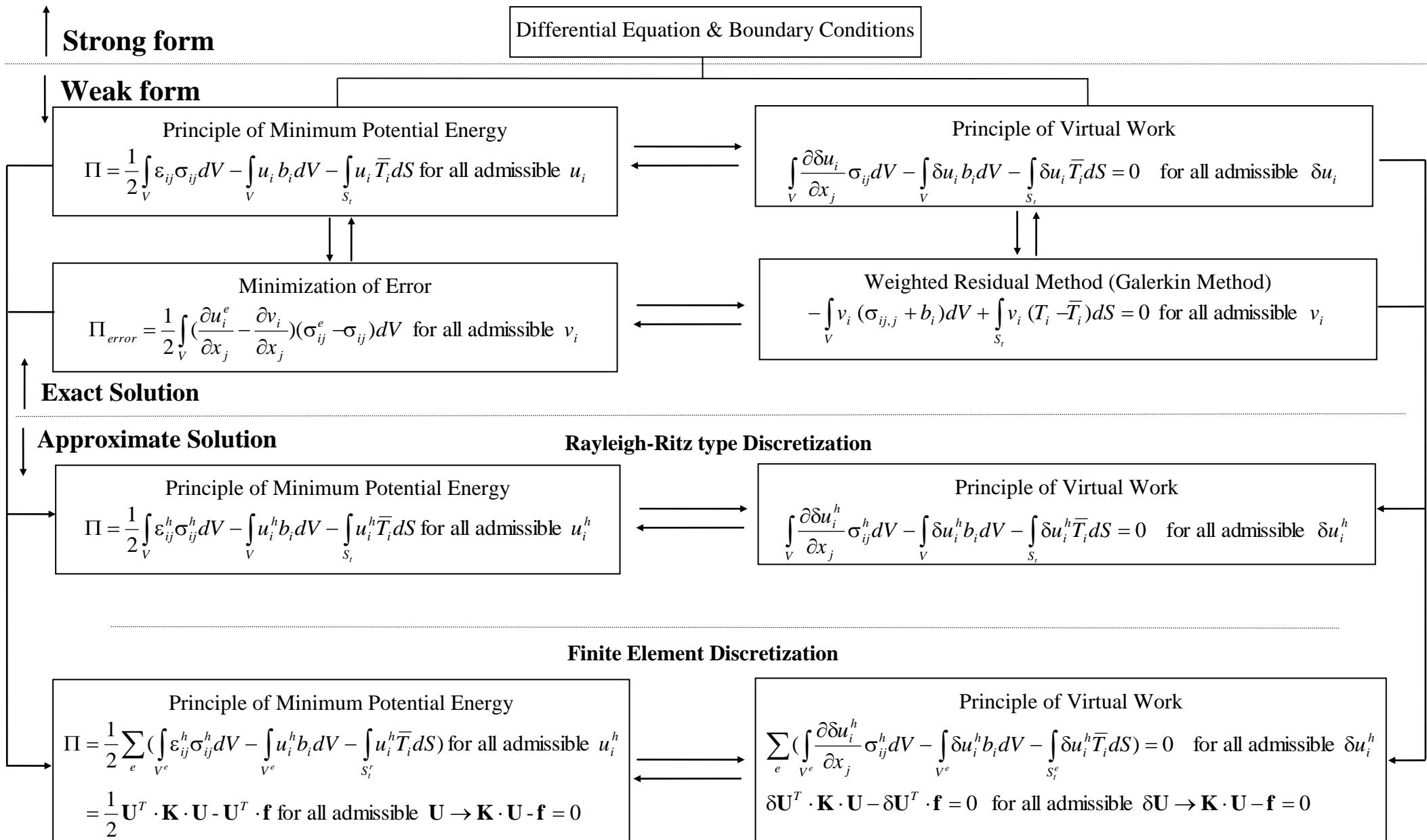
Finite Element Programming

(Linear Static case)



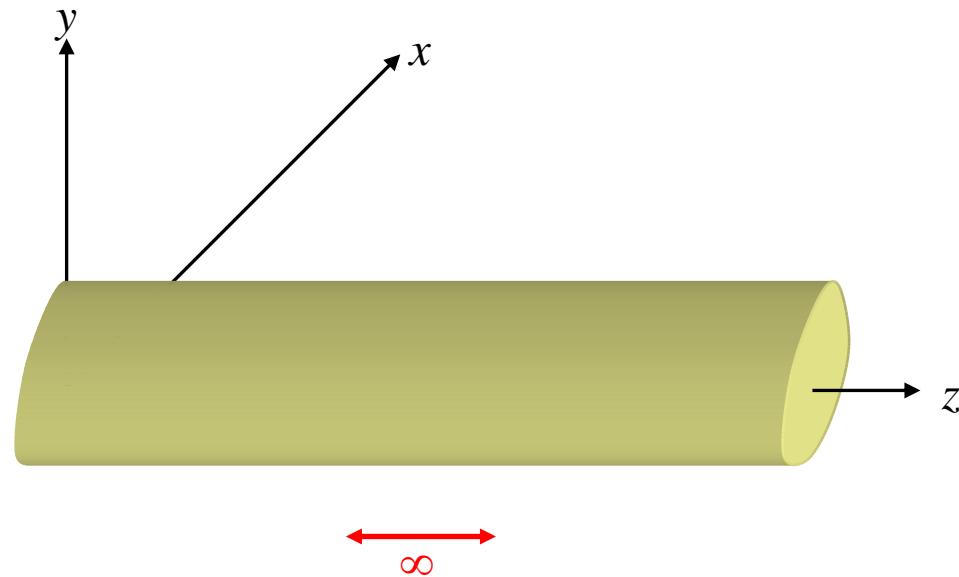
- **Data Structure**

- Control Data : # of nodes, # of elements, # of support, # of forces applied at nodes ...
- Geometry Data : Nodal Coordinates & Element information (Type, Material Properties, Incidences)
- Material Properties
- Boundary Condition : Traction BC & Displacement BC
- Miscellaneous options



Chapter 6.

Two-dimensional Elasticity Problems



Plane Stress

- **Stress** : $\sigma_{33} = \sigma_{13} = \sigma_{23} = 0$
- **Strain** : $\varepsilon_{33} = -\frac{\nu}{1-\nu}(\varepsilon_{11} + \varepsilon_{22})$, $\gamma_{13} = 0$, $\gamma_{23} = 0$
- **Modified Stress-strain Relation**

$$\sigma_{11} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}(\varepsilon_{11} + \frac{\nu}{1-\nu}(\varepsilon_{22} + \varepsilon_{33})) = \frac{E}{1-\nu^2}(\varepsilon_{11} + \nu\varepsilon_{22})$$

$$\sigma_{22} = \frac{E}{1-\nu^2}(\nu\varepsilon_{11} + \varepsilon_{22}), \quad \sigma_{12} = \frac{E}{2(1+\nu)}\gamma_{12}$$

$$\boldsymbol{\sigma}^e = \begin{pmatrix} \sigma_{11}^e \\ \sigma_{22}^e \\ \sigma_{12}^e \end{pmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{pmatrix} \varepsilon_{11}^e \\ \varepsilon_{22}^e \\ \gamma_{12}^e \end{pmatrix} = \mathbf{D}\boldsymbol{\varepsilon}^e$$

- **Interpolation of Displacement :** $\mathbf{u}^e = \begin{pmatrix} u^e \\ v^e \end{pmatrix} = \begin{bmatrix} N_1 & 0 & \dots & N_n & 0 \\ 0 & N_1 & & 0 & N_n \end{bmatrix} \begin{pmatrix} u_1 \\ v_1 \\ \vdots \\ u_n \\ v_n \end{pmatrix}^e = \mathbf{N}^e \mathbf{U}^e$
- **Strain-Displacement Relation**

$$\boldsymbol{\epsilon}^e = \begin{pmatrix} \epsilon_{11}^e \\ \epsilon_{22}^e \\ \gamma_{12}^e \end{pmatrix} = \begin{pmatrix} \frac{\partial u^e}{\partial x} \\ \frac{\partial v^e}{\partial y} \\ \frac{\partial u^e}{\partial y} + \frac{\partial v^e}{\partial x} \end{pmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_n}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & \dots & 0 & \frac{\partial N_n}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_n}{\partial y} & \frac{\partial N_n}{\partial x} \end{bmatrix} \begin{pmatrix} u_1 \\ v_1 \\ \vdots \\ u_n \\ v_n \end{pmatrix}^e = \mathbf{B} \mathbf{U}^e$$

- **Principle of Virtual work**

$$\int_A (\delta \epsilon_{11}^h \sigma_{11}^h + \delta \epsilon_{22}^h \sigma_{22}^h + \delta \gamma_{12}^h \sigma_{12}^h) t dA = \int_A (\delta u^h b_x + \delta v^h b_y) t dA + \int_{S_t} (\delta u^h T_x + \delta v^h T_y) t dS$$

$$\sum_e \int_{A^e} \mathbf{B}^T \cdot \mathbf{D} \cdot \mathbf{B} t dA \mathbf{U} = \sum_e \int_{A^e} \mathbf{N}^T \cdot \mathbf{b} t dA + \sum_e \int_{S^e} \mathbf{N}^T \cdot \mathbf{T} t dS$$

Plane Strain

- **Strain** : $\varepsilon_{33} = 0, \gamma_{13} = 0, \gamma_{23} = 0$
- **Stress** : $\sigma_{13} = \sigma_{23} = 0, \sigma_{33} = -\nu(\sigma_{11} + \sigma_{22})$
- **Stress-strain Relation**

$$\boldsymbol{\sigma}^e = \begin{pmatrix} \sigma_{11}^e \\ \sigma_{22}^e \\ \sigma_{12}^e \end{pmatrix} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & 0 \\ \frac{\nu}{1-\nu} & 1 & 0 \\ 0 & 0 & \frac{1-2\nu}{2(1-\nu)} \end{bmatrix} \begin{pmatrix} \varepsilon_{11}^e \\ \varepsilon_{22}^e \\ \gamma_{12}^e \end{pmatrix} = \mathbf{D}\boldsymbol{\varepsilon}^e$$

- **Interpolation of Displacement**

$$\mathbf{u}^e = \begin{pmatrix} u^e \\ v^e \end{pmatrix} = \begin{bmatrix} N_1 & 0 & \dots & N_n & 0 \\ 0 & N_1 & \dots & 0 & N_n \end{bmatrix} \begin{pmatrix} u_1 \\ v_1 \\ \vdots \\ u_n \\ v_n \end{pmatrix}^e = \mathbf{N}^e \mathbf{U}^e$$

- **Strain-Displacement Relation**

$$\boldsymbol{\varepsilon}^e = \begin{pmatrix} \varepsilon_{11}^e \\ \varepsilon_{22}^e \\ \gamma_{12}^e \end{pmatrix} = \begin{pmatrix} \frac{\partial u^e}{\partial x} \\ \frac{\partial v^e}{\partial y} \\ \frac{\partial u^e}{\partial y} + \frac{\partial v^e}{\partial x} \end{pmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_n}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & \dots & 0 & \frac{\partial N_n}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_n}{\partial y} & \frac{\partial N_n}{\partial x} \end{bmatrix} \begin{pmatrix} u_1 \\ v_1 \\ \vdots \\ u_n \\ v_n \end{pmatrix}^e = \mathbf{B}\mathbf{U}^e$$

- **Principle of Virtual work**

$$\int_A (\delta\varepsilon_{11}^h \sigma_{11}^h + \delta\varepsilon_{22}^h \sigma_{22}^h + \delta\gamma_{12}^h \sigma_{12}^h) t dA = \int_A (\delta u^h b_x + \delta v^h b_y) t dA + \int_{S_t} (\delta u^h T_x + \delta v^h T_y) t dS$$

$$\sum_e \int_{A^e} \mathbf{B}^T \cdot \mathbf{D} \cdot \mathbf{B} t dA \mathbf{U} = \sum_e \int_{A^e} \mathbf{N}^T \cdot \mathbf{b} t dA + \sum_e \int_{S^e} \mathbf{N}^T \cdot \mathbf{T} t dS$$

Axisymmetry

- **Strain :** $\varepsilon_{rr} = \frac{\partial u}{\partial r}$, $\varepsilon_{zz} = \frac{\partial v}{\partial z}$, $\varepsilon_{\theta\theta} = \frac{u}{r}$, $\gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial v}{\partial r}$
 $\gamma_{r\theta} = 0$, $\gamma_{z\theta} = 0$
- **Stress :** $\sigma_{r\theta} = \sigma_{z\theta} = 0$
- **Stress-strain Relation**

$$\boldsymbol{\sigma}^e = \begin{pmatrix} \sigma_{rr}^e \\ \sigma_{zz}^e \\ \sigma_{\theta\theta}^e \\ \sigma_{rz}^e \end{pmatrix} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 0 \\ \frac{\nu}{1-\nu} & 1 & \frac{\nu}{1-\nu} & 0 \\ \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 1 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} \end{bmatrix} \begin{pmatrix} \varepsilon_{rr}^e \\ \varepsilon_{zz}^e \\ \varepsilon_{\theta\theta}^e \\ \gamma_{rz}^e \end{pmatrix} = \mathbf{D}\boldsymbol{\varepsilon}^e$$

- **Interpolation of Displacement**

$$\mathbf{u}^e = \begin{pmatrix} u^e \\ v^e \end{pmatrix} = \begin{bmatrix} N_1 & 0 & \dots & N_n & 0 \\ 0 & N_1 & & 0 & N_n \end{bmatrix} \begin{pmatrix} u_1 \\ v_1 \\ \vdots \\ u_n \\ v_n \end{pmatrix}^e = \mathbf{N}^e \mathbf{U}^e$$

- **Strain-Displacement Relation**

$$\boldsymbol{\varepsilon}^e = \begin{pmatrix} \varepsilon_{rr}^e \\ \varepsilon_{zz}^e \\ \varepsilon_{\theta\theta}^e \\ \gamma_{rz}^e \end{pmatrix} = \begin{pmatrix} \frac{\partial u^e}{\partial r} \\ \frac{\partial v^e}{\partial z} \\ \frac{u^e}{r} \\ \frac{\partial u^e}{\partial z} + \frac{\partial v^e}{\partial r} \end{pmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial r} & 0 & \frac{\partial N_n}{\partial r} & 0 \\ 0 & \frac{\partial N_1}{\partial z} & \dots & \frac{\partial N_n}{\partial z} \\ \frac{N_1}{r} & 0 & \frac{N_n}{r} & 0 \\ \frac{\partial N_1}{\partial z} & \frac{\partial N_1}{\partial r} & \frac{\partial N_n}{\partial z} & \frac{\partial N_n}{\partial r} \end{bmatrix} \begin{pmatrix} u_1 \\ v_1 \\ \vdots \\ u_n \\ v_n \end{pmatrix}^e = \mathbf{B} \mathbf{U}^e$$

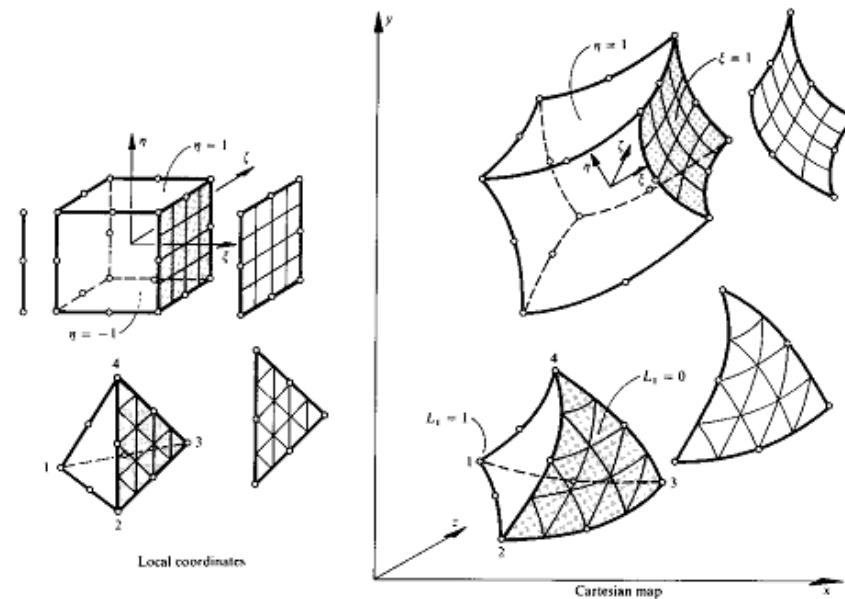
- Principle of Virtual work

$$\int_A (\delta \varepsilon_{rr}^h \sigma_{rr}^h + \delta \varepsilon_{zz}^h \sigma_{zz}^h + \delta \varepsilon_{\theta\theta}^h \sigma_{\theta\theta}^h + \delta \gamma_{r\theta}^h \sigma_{r\theta}^h) 2\pi r dA = \int_A (\delta u^h b_r + \delta v^h b_z) 2\pi r dA + \int_{S_t} (\delta u^h T_r + \delta v^h T_z) 2\pi r dS$$

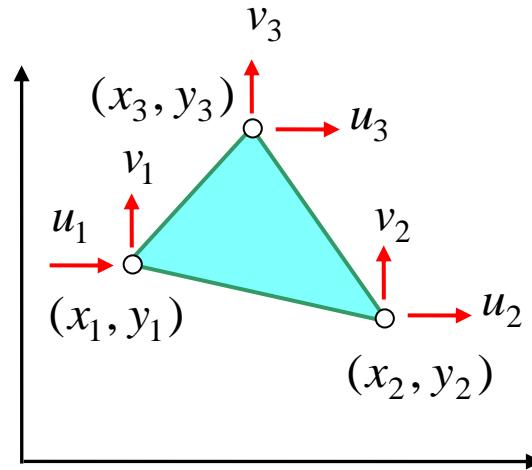
$$\sum_e \int_{A^e} \mathbf{B}^T \cdot \mathbf{D} \cdot \mathbf{B} 2\pi r dA \mathbf{U} = \sum_e \int_{A^e} \mathbf{N}^T \cdot \mathbf{b} 2\pi r dA + \sum_e \int_{S^e} \mathbf{N}^T \cdot \mathbf{T} 2\pi r dS$$

Chapter 7

Various Types of Elements



Constant Strain Triangle (CST) Element



$$u^e(x, y) = \alpha_1 + \alpha_2 x + \alpha_3 y, \quad v^e(x, y) = \alpha_4 + \alpha_5 x + \alpha_6 y$$

$$\begin{aligned} u^e(x_1, y_1) &= u_1 = \alpha_1 + \alpha_2 x_1 + \alpha_3 y_1 \\ u^e(x_2, y_2) &= u_2 = \alpha_1 + \alpha_2 x_2 + \alpha_3 y_2 \rightarrow \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \\ u^e(x_3, y_3) &= u_3 = \alpha_1 + \alpha_2 x_3 + \alpha_3 y_3 \end{aligned}$$

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \frac{1}{2\Delta} \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

where $a_i = x_j y_m - x_m y_j$, $b_i = y_j - y_m$, $c_i = x_m - x_j$

$$u^e(x, y) = \frac{1}{2\Delta} (a_1 + b_1 x + c_1 y) u_1 + \frac{1}{2\Delta} (a_2 + b_2 x + c_2 y) u_2 + \frac{1}{2\Delta} (a_3 + b_3 x + c_3 y) u_3$$

$$v^e(x, y) = \frac{1}{2\Delta} (a_1 + b_1 x + c_1 y) v_1 + \frac{1}{2\Delta} (a_2 + b_2 x + c_2 y) v_2 + \frac{1}{2\Delta} (a_3 + b_3 x + c_3 y) v_3$$

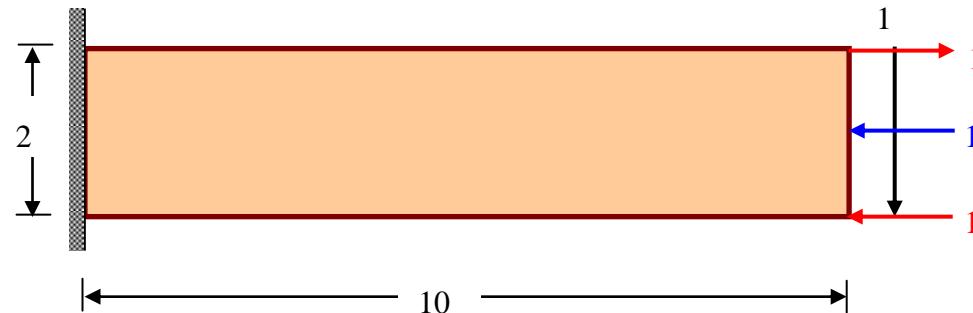
$$\boldsymbol{\epsilon}^e = \begin{pmatrix} \epsilon_{11}^e \\ \epsilon_{22}^e \\ \gamma_{12}^e \end{pmatrix} = \begin{pmatrix} \frac{\partial u^e}{\partial x} \\ \frac{\partial v^e}{\partial y} \\ \frac{\partial u^e}{\partial y} + \frac{\partial v^e}{\partial x} \end{pmatrix} = \frac{1}{2\Delta} \begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{pmatrix}^e = \mathbf{B} \mathbf{U}^e$$

$$\mathbf{K}^e = \int_{A^e} \mathbf{B}^T \cdot \mathbf{D} \cdot \mathbf{B} t dA \mathbf{U} = \mathbf{B}^T \cdot \mathbf{D} \cdot \mathbf{B} t \Delta^e$$

Homework 6

- Cantilever Beam with three load cases -

Implement your own finite element program for 2-D elasticity problems using CST element. When you build your program, consider expandability so that you can easily add other types of elements to your program for next homeworks.



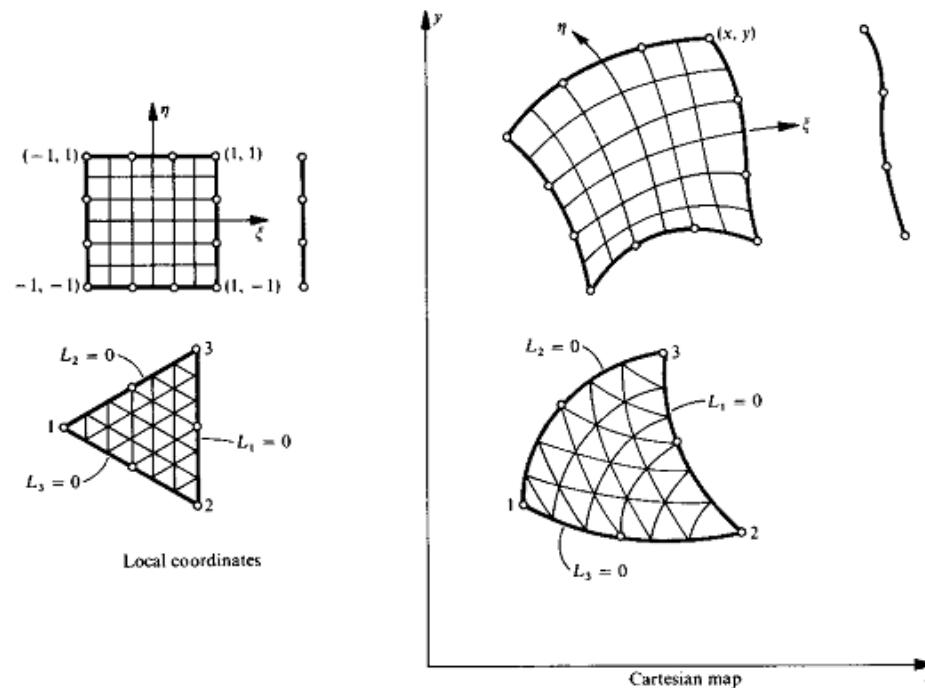
- a) Discuss how to simulate two boundary conditions given in the Timoshenko's book (equation(k) and (l) in page 44).
- b) For end shear load case, solve the problem for both boundary conditions with 40 CST elements.
- c) For other load cases, use one boundary condition of your choice with 40 CST elements
- d) Perform the convergence test with at least 5 different mesh layouts for the end shear load case. Use the boundary condition of your choice.
- e) Discuss local effects, St-Venant effect, Poisson effect stress concentration, etc. Present suitable plots and tables of displacement and stress to justify or clarify your discussions.
- f) Comparison of your results with other solutions such as analytic solutions, one-dimensional solutions is strongly recommended for your discussion . Assume $E = 1.0$, $\nu = 0.3$.

Isoparametric Formulation

- **Interpolation of Geometry**

$$x = \bar{N}_1(\xi, \eta)x_1 + \cdots + \bar{N}_m(\xi, \eta)x_m = \sum_{i=1}^m \bar{N}_i(\xi, \eta)x_i$$

$$y = \bar{N}_1(\xi, \eta)y_1 + \cdots + \bar{N}_m(\xi, \eta)y_m = \sum_{i=1}^m \bar{N}_i(\xi, \eta)y_i$$



$$\mathbf{x}^e = \begin{pmatrix} x^e \\ y^e \end{pmatrix} = \begin{bmatrix} \bar{N}_1 & 0 & \bar{N}_2 & 0 & \dots & \bar{N}_m & 0 \\ 0 & \bar{N}_1 & 0 & \bar{N}_2 & \dots & 0 & \bar{N}_m \end{bmatrix} \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ \vdots \\ x_m \\ y_m \end{pmatrix}^e = \bar{\mathbf{N}}^e \mathbf{X}^e$$

- **Interpolation of Displacement in a Parent Element**

$$\mathbf{u}^e = \begin{pmatrix} u^e \\ v^e \end{pmatrix} = \mathbf{N}^e(\xi, \eta) \mathbf{U}^e$$

- **Derivatives of the Displacement Shape Functions**

$$\left. \begin{aligned} \frac{\partial N_i}{\partial \xi} &= \frac{\partial N_i}{\partial x} \frac{\partial x}{\partial \xi} + \frac{\partial N_i}{\partial y} \frac{\partial y}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} &= \frac{\partial N_i}{\partial x} \frac{\partial x}{\partial \eta} + \frac{\partial N_i}{\partial y} \frac{\partial y}{\partial \eta} \end{aligned} \right\} \rightarrow \begin{pmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{pmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{pmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{pmatrix} = \mathbf{J}^{-1} \begin{pmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{pmatrix} \text{ or } \nabla_x N_i = \mathbf{J}^{-1} \nabla_{\eta} N_i$$

$$\mathbf{J} = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^m \frac{\partial \bar{N}_i}{\partial \xi} x_i & \sum_{i=1}^m \frac{\partial \bar{N}_i}{\partial \xi} y_i \\ \sum_{i=1}^m \frac{\partial \bar{N}_i}{\partial \eta} x_i & \sum_{i=1}^m \frac{\partial \bar{N}_i}{\partial \eta} y_i \end{pmatrix} = \begin{pmatrix} \frac{\partial \bar{N}_1}{\partial \xi} & \frac{\partial \bar{N}_2}{\partial \xi} & \dots & \frac{\partial \bar{N}_m}{\partial \xi} \\ \frac{\partial \bar{N}_1}{\partial \eta} & \frac{\partial \bar{N}_2}{\partial \eta} & \dots & \frac{\partial \bar{N}_m}{\partial \eta} \end{pmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots \\ x_m & y_m \end{bmatrix} = \nabla_{\xi} \bar{\mathbf{N}} \cdot \mathbf{X}$$

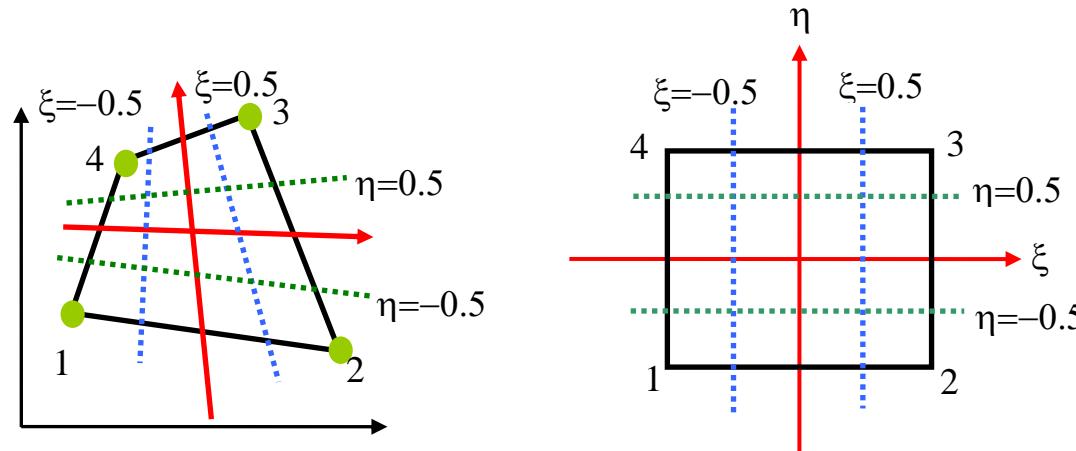
$m > n \quad N \neq \bar{N}$: Superparametric element

$m = n \quad N = \bar{N}$: **Isoparametric element**

$m < n \quad N \neq \bar{N}$: Subparametric element

$$\int_{A^e} \mathbf{B}^T \cdot \mathbf{D} \cdot \mathbf{B} t dA = \int_{-1}^1 \int_{-1}^1 \mathbf{B}^T(\xi, \eta) \cdot \mathbf{D} \cdot \mathbf{B}(\xi, \eta) t | J | d\xi d\eta$$

Bilinear Isoparametric Element



- Shape functions in the parent coordinate system.

$$u(x(\xi, \eta), y(\xi, \eta)) = \alpha_1 + \alpha_2\xi + \alpha_3\eta + \alpha_4\xi\eta$$

$$v(x(\xi, \eta), y(\xi, \eta)) = \alpha_5 + \alpha_6\xi + \alpha_7\eta + \alpha_8\xi\eta$$

$$u_1 = u(x_1, y_1) = u(x(\xi_1, \eta_1), y(\xi_1, \eta_1)) = \alpha_1 + \alpha_2\xi_1 + \alpha_3\eta_1 + \alpha_4\xi_1\eta_1 = \alpha_1 - \alpha_2 - \alpha_3 + \alpha_4$$

$$u_2 = u(x_2, y_2) = u(x(\xi_2, \eta_2), y(\xi_2, \eta_2)) = \alpha_1 + \alpha_2\xi_2 + \alpha_3\eta_2 + \alpha_4\xi_2\eta_2 = \alpha_1 + \alpha_2 - \alpha_3 - \alpha_4$$

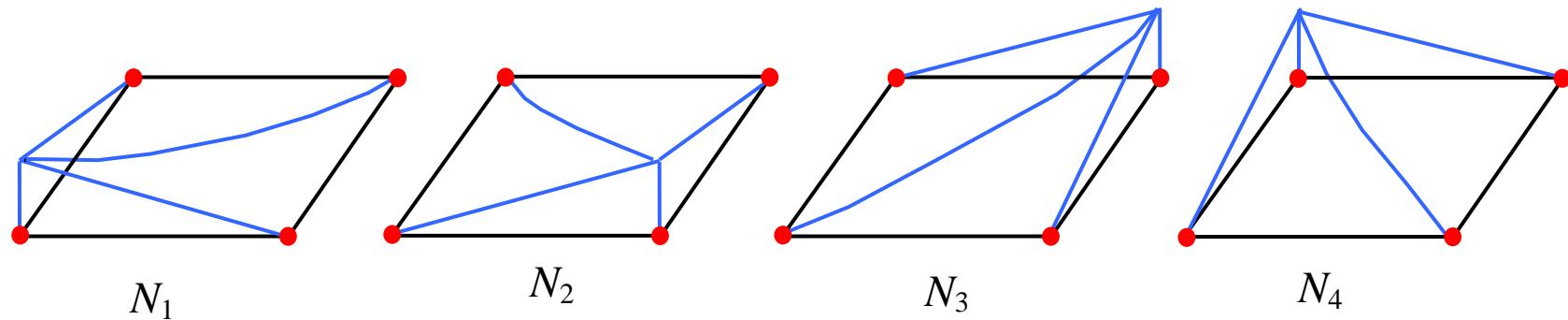
$$u_3 = u(x_3, y_3) = u(x(\xi_3, \eta_3), y(\xi_3, \eta_3)) = \alpha_1 + \alpha_2\xi_3 + \alpha_3\eta_3 + \alpha_4\xi_3\eta_3 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$$

$$u_4 = u(x_4, y_4) = u(x(\xi_4, \eta_4), y(\xi_4, \eta_4)) = \alpha_1 + \alpha_2\xi_4 + \alpha_3\eta_4 + \alpha_4\xi_4\eta_4 = \alpha_1 - \alpha_2 + \alpha_3 - \alpha_4$$

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix}$$

$$u^e(x, y) = N_1 u^1 + N_2 u^2 + N_3 u^3 + N_4 u^4, \quad v^e(x, y) = N_1 v^1 + N_2 v^2 + N_3 v^3 + N_4 v^4$$

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta), \quad N_2 = \frac{1}{4}(1+\xi)(1-\eta), \quad N_3 = \frac{1}{4}(1+\xi)(1+\eta), \quad N_4 = \frac{1}{4}(1-\xi)(1+\eta)$$



Higher Order Rectangular Element

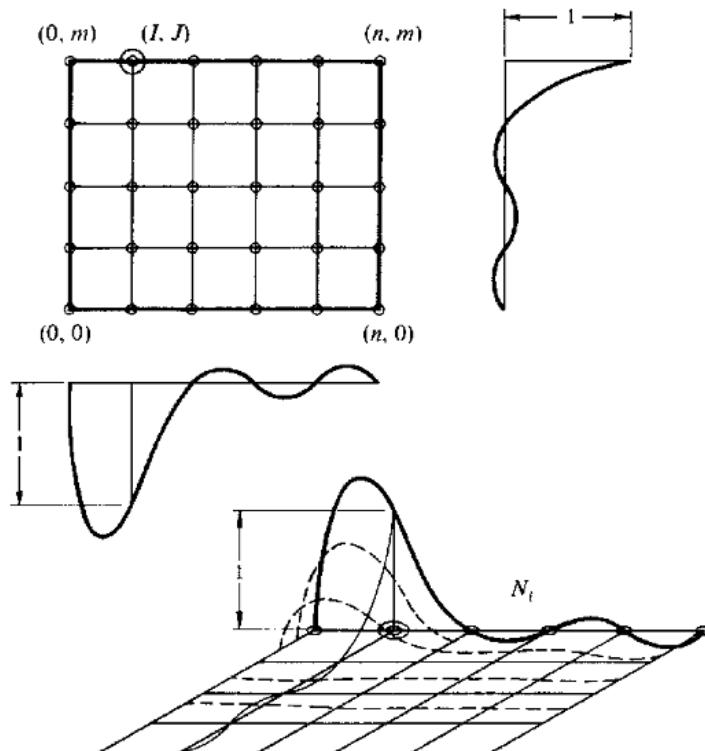
-Lagrange Family-

- Shape function of m-th order for k-th node in one dimension

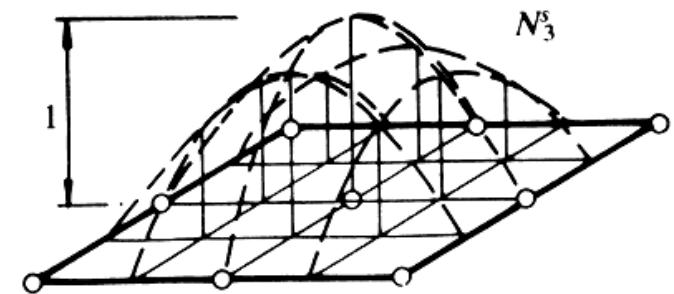
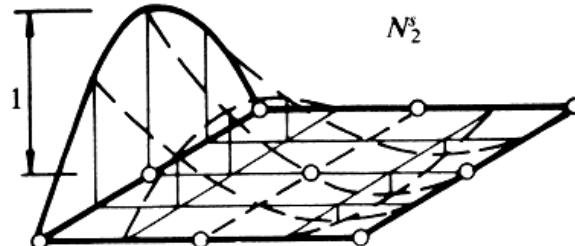
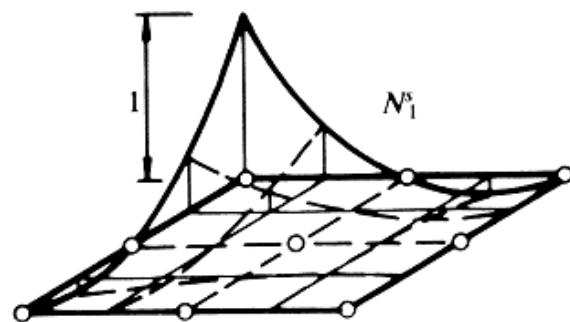
$$l_k^m(\xi) = \frac{(\xi - \xi_1) \cdots (\xi - \xi_{k-1})(\xi - \xi_{k+1}) \cdots (\xi - \xi_{m+1})}{(\xi_k - \xi_1) \cdots (\xi_k - \xi_{k-1})(\xi_k - \xi_{k+1}) \cdots (\xi_k - \xi_{m+1})}$$

$$l_k^m(\eta) = \frac{(\eta - \eta_1) \cdots (\eta - \eta_{k-1})(\eta - \eta_{k+1}) \cdots (\eta - \eta_{m+1})}{(\eta_k - \eta_1) \cdots (\eta_k - \eta_{k-1})(\eta_k - \eta_{k+1}) \cdots (\eta_k - \eta_{m+1})}$$

$$N_i = N_{IJ} = l_I^m(\xi) \times l_J^m(\eta)$$

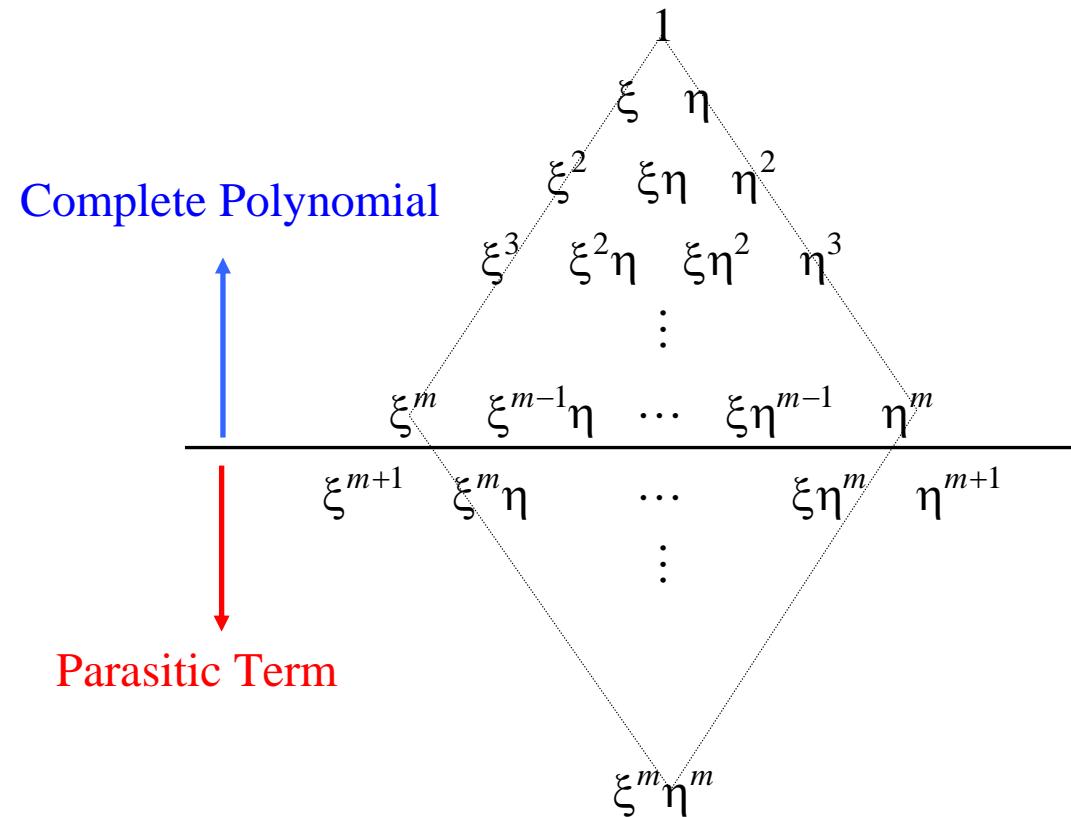


- **Q9 Element**



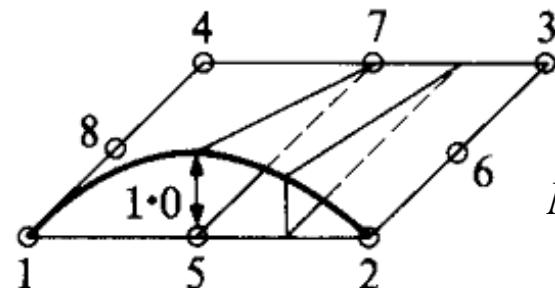
- Total number of nodes in an element : $(m+1)(m+1)$
- Total number of terms in m-th order polynomials : $\frac{(m+2)(m+1)}{2}$
- Total number of the parasitic terms : $\frac{(m+1)m}{2}$

- The Pascal Polynomials

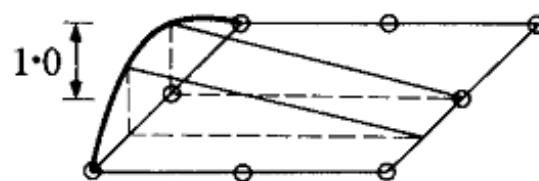


Higher Order Rectangular Element -Serendipity Family-

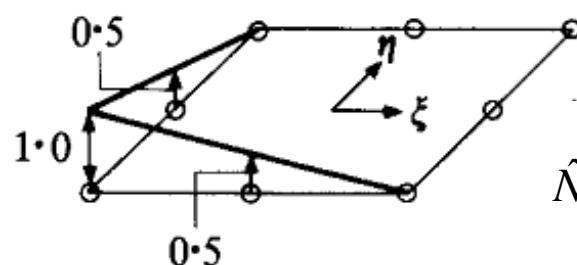
- Q8 Element



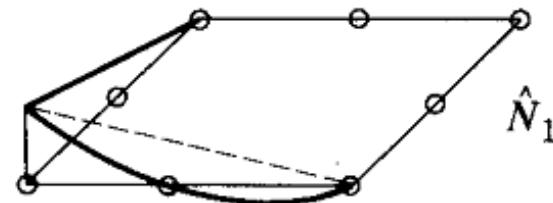
$$N_5 = \frac{1}{2}(1 - \xi^2)(1 - \eta)$$



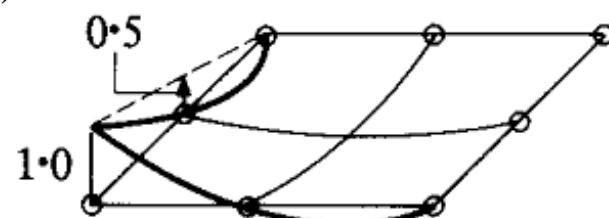
$$N_8 = \frac{1}{2}(1 - \xi)(1 - \eta^2)$$



$$\hat{N}_1 = \frac{1}{4}(1 - \xi)(1 - \eta)$$



$$\hat{N}_1 = \frac{1}{2} N_5$$



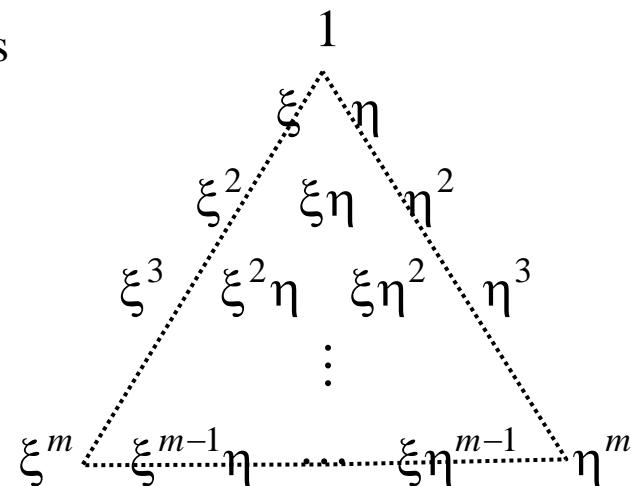
$$N_{\hat{N}_1} = \hat{N}_1 - \frac{1}{2} N_5 - \frac{1}{2} N_8$$

Triangular Isoparametric Element

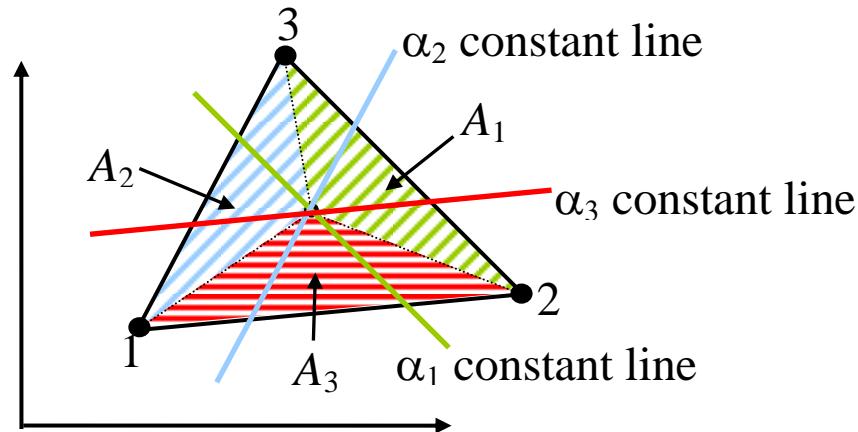
- Total number of nodes on sides of a triangle element for m -th order S.F.: $3m$
- Total number of terms in m -th order polynomials :

$$\frac{(m+2)(m+1)}{2}$$
- Total number of internal nodes :

$$\frac{(m+2)(m+1)}{2} - 3m = \frac{(m-2)(m-1)}{2}$$
- The Pascal Polynomials



- Area Coordinate System



$$\alpha_1 = \frac{A_1}{A}, \quad \alpha_2 = \frac{A_2}{A}, \quad \alpha_3 = \frac{A_3}{A}, \quad \alpha_1 + \alpha_2 + \alpha_3 = 1$$

- Shape functions

- CST Element

$$N_1 = \alpha_1, \quad N_2 = \alpha_2, \quad N_3 = \alpha_3$$

- LST Element

$$\begin{aligned} N_1 &= \alpha_1(2\alpha_1 - 1) & N_2 &= \alpha_2(2\alpha_2 - 1) & N_3 &= \alpha_3(2\alpha_3 - 1) \\ N_4 &= 4\alpha_1\alpha_2 & N_5 &= 4\alpha_2\alpha_3 & N_6 &= 4\alpha_1\alpha_3 \end{aligned}$$

- Interpolation of Geometry

$$x = \sum_{i=1}^n N_i(\alpha_1, \alpha_2, \alpha_3) x_i = \sum_{i=1}^n \tilde{N}_i(\alpha_1, \alpha_2) x_i$$

$$y = \sum_{i=1}^n N_i(\alpha_1, \alpha_2, \alpha_3) y_i = \sum_{i=1}^n \tilde{N}_i(\alpha_1, \alpha_2) y_i$$

$$(\mathbf{X}^e) = \begin{pmatrix} x^e \\ y^e \end{pmatrix} = \begin{bmatrix} \tilde{N}_1 & 0 & \dots & \tilde{N}_n & 0 \\ 0 & \tilde{N}_1 & 0 & \dots & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ y_1 \\ \vdots \\ x_m \\ y_m \end{pmatrix}^e = [\tilde{N}]^e (X^e)$$

- Interpolation of Displacement in a Parent Element

$$(u^e) = \begin{pmatrix} u^e \\ v^e \end{pmatrix} = [N(\alpha_1, \alpha_2, \alpha_3)]^e (U^e) = [\tilde{N}(\alpha_1, \alpha_2)]^e (U^e)$$

- Derivatives of the Displacement Shape Functions

$$\left. \begin{aligned} \frac{\partial \tilde{N}_i}{\partial \alpha_1} &= \frac{\partial \tilde{N}_i}{\partial x} \frac{\partial x}{\partial \alpha_1} + \frac{\partial \tilde{N}_i}{\partial y} \frac{\partial y}{\partial \alpha_1} \\ \frac{\partial \tilde{N}_i}{\partial \alpha_2} &= \frac{\partial \tilde{N}_i}{\partial x} \frac{\partial x}{\partial \alpha_2} + \frac{\partial \tilde{N}_i}{\partial y} \frac{\partial y}{\partial \alpha_2} \end{aligned} \right\} \rightarrow \begin{pmatrix} \frac{\partial \tilde{N}_i}{\partial \alpha_1} \\ \frac{\partial \tilde{N}_i}{\partial \alpha_2} \end{pmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \alpha_1} & \frac{\partial y}{\partial \alpha_1} \\ \frac{\partial x}{\partial \alpha_2} & \frac{\partial y}{\partial \alpha_2} \end{bmatrix} \begin{pmatrix} \frac{\partial \tilde{N}_i}{\partial x} \\ \frac{\partial \tilde{N}_i}{\partial y} \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} \frac{\partial \tilde{N}_i}{\partial x} \\ \frac{\partial \tilde{N}_i}{\partial y} \end{pmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \alpha_1} & \frac{\partial y}{\partial \alpha_1} \\ \frac{\partial x}{\partial \alpha_2} & \frac{\partial y}{\partial \alpha_2} \end{bmatrix}^{-1} \begin{pmatrix} \frac{\partial \tilde{N}_i}{\partial \alpha_1} \\ \frac{\partial \tilde{N}_i}{\partial \alpha_2} \end{pmatrix} = \mathbf{J}^{-1} \begin{pmatrix} \frac{\partial \tilde{N}_i}{\partial \alpha_1} \\ \frac{\partial \tilde{N}_i}{\partial \alpha_2} \end{pmatrix}^i$$

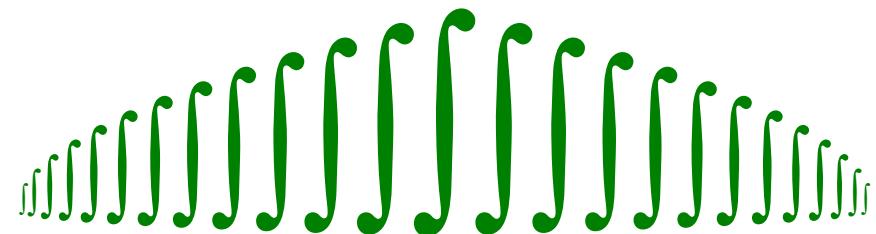
or $\nabla_x \tilde{N}_i = \mathbf{J}^{-1} \nabla_\alpha \tilde{N}_i$

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial \alpha_1} & \frac{\partial y}{\partial \alpha_1} \\ \frac{\partial x}{\partial \alpha_2} & \frac{\partial y}{\partial \alpha_2} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^m \frac{\partial \tilde{N}_i}{\partial \alpha_1} x_i & \sum_{i=1}^m \frac{\partial \tilde{N}_i}{\partial \alpha_1} y_i \\ \sum_{i=1}^m \frac{\partial \tilde{N}_i}{\partial \alpha_2} x_i & \sum_{i=1}^m \frac{\partial \tilde{N}_i}{\partial \alpha_2} y_i \end{bmatrix} = \begin{bmatrix} \frac{\partial \tilde{N}_1}{\partial \alpha_1} & \frac{\partial \tilde{N}_2}{\partial \alpha_1} & \dots & \frac{\partial \tilde{N}_m}{\partial \alpha_1} \\ \frac{\partial \tilde{N}_1}{\partial \alpha_2} & \frac{\partial \tilde{N}_2}{\partial \alpha_2} & \dots & \frac{\partial \tilde{N}_m}{\partial \alpha_2} \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots \\ x_m & y_m \end{bmatrix} = \nabla_\alpha \tilde{\mathbf{N}} \cdot \mathbf{X}$$

Homework 7

Chapter 8

Numerical Integration



8.1. Gauss Quadrature Rule

- One Dimension

$$\int_{-1}^1 f(\xi) d\xi \approx \sum_{i=1}^n W_i f(\xi_i)$$

If the given function $f(\xi)$ is a polynomial, it is possible to construct the quadrature rule that yields the exact integration.

- $f(\xi)$ is constant: $f(\xi) = a_0$

$$\int_{-1}^1 f(\xi) dx = 2a_0 = a_0 \sum_{i=1}^n W_i \rightarrow n=1 \quad W_1 = 2$$

- $f(\xi)$ is first order: $f(\xi) = a_0 + a_1 \xi$ One point rule is good enough.

- $f(\xi)$ is second order: $f(\xi) = a_0 + a_1 \xi + a_2 \xi^2$

$$\int_{-1}^1 f(\xi) = \frac{2}{3} a_2 + 2a_0 = a_2 \sum_{i=1}^n W_i \xi_i^2 + a_1 \sum_{i=1}^n W_i \xi_i + a_0 \sum_{i=1}^n W_i \rightarrow$$

$$\sum_{i=1}^n W_i \xi_i^2 = \frac{2}{3}, \quad \sum_{i=1}^n W_i \xi_i = 0, \quad \sum_{i=1}^n W_i = 2 \rightarrow n = 2$$

$$\sum_{i=1}^n W_i \xi_i^1 = 0 \rightarrow W_1 = W_2 , \quad \xi_1 = -\xi_2$$

$$\sum_{i=1}^2 W_i \xi_i^2 = W_1 \xi_1^2 + W_2 \xi_2^2 = 2W_2 \xi_2^2 = \frac{2}{3} \rightarrow W_2 \xi_2^2 = \frac{1}{3}$$

$$\sum_{i=1}^2 W_i = W_1 + W_2 = 2W_2 = 2 \rightarrow W_2 = 1 \rightarrow \xi_2 = \sqrt{1/3} = 0.57735\ 02691\ 89626$$

- $f(\xi)$ is third order: $f(\xi) = a_0 + a_1 \xi + a_2 \xi^2 + a_3 \xi^3$ Two point rule is enough.

- $f(\xi)$ is fourth order: $f(\xi) = a_0 + a_1 \xi + a_2 \xi^2 + a_3 \xi^3 + a_4 \xi^4$

$$\int_{-1}^1 f(\xi) d\xi = \frac{2}{5} a_4 + \frac{2}{3} a_2 + 2a_0 = a_4 \sum_{i=1}^n W_i \xi_i^4 + a_3 \sum_{i=1}^n W_i \xi_i^3 + a_2 \sum_{i=1}^n W_i \xi_i^2 + a_1 \sum_{i=1}^n W_i \xi_i + a_0 \sum_{i=1}^n W_i \rightarrow$$

$$\sum_{i=1}^n W_i \xi_i^4 = \frac{2}{5}, \quad \sum_{i=1}^n W_i \xi_i^3 = 0, \quad \sum_{i=1}^n W_i \xi_i^2 = \frac{2}{3}, \quad \sum_{i=1}^n W_i \xi_i = 0, \quad \sum_{i=1}^n W_i = 2 \rightarrow n = 3$$

$$\sum_{i=1}^n W_i \xi_i^3 = 0, \quad \sum_{i=1}^n W_i \xi_i = 0 \rightarrow W_1 = W_3, \quad \xi_1 = -\xi_3, \quad \xi_2 = 0$$

$$\sum_{i=1}^3 W_i \xi_i^4 = W_1 \xi_1^4 + W_3 \xi_3^4 = 2W_3 \xi_3^4 = \frac{2}{5} \rightarrow W_3 \xi_3^4 = \frac{1}{5}$$

$$\sum_{i=1}^3 W_i \xi_i^2 = W_1 \xi_1^2 + W_3 \xi_3^2 = 2W_3 \xi_3^2 = \frac{2}{3} \rightarrow W_3 \xi_3^2 = \frac{1}{3} \rightarrow \xi_3 = \sqrt{3/5}, W_3 = \frac{5}{9}$$

$$\xi_3 = 0.77459\ 66692\ 41483, W_3 = 0.55555\ 55555\ 55555$$

$$\sum_{i=1}^3 W_i = W_1 + W_2 + W_3 = 2W_3 + W_2 = 2 \rightarrow W_2 = \frac{8}{9} = 0.88888\ 88888\ 88888$$

- Because of the symmetry condition, we need to decide only n unknowns for n -points G.Q..
- We can integrate $2n-1$ -th polynomials exactly with n -points G.Q. Since for $2m$ -th order polynomials we have $2m$ conditions for G.Q. -which means we can determine $(m+1)$ -point G.Q..
- Stiffness Equation

$$\int_{V^e} \mathbf{B}^T \cdot \mathbf{D} \cdot \mathbf{B} dV = \int_{x^e} \mathbf{B}^T \cdot \mathbf{D} \cdot \mathbf{B} A dx = \int_{-1}^1 \mathbf{B}^T(\xi) \cdot \mathbf{D} \cdot \mathbf{B}(\xi) A |J| d\xi = \sum_{i=1}^n W_i \mathbf{B}^T(\xi_i) \cdot \mathbf{D}_i \cdot \mathbf{B}(\xi_i) A_i |J_i|$$

$$\int_{V^e} \mathbf{N}^T \cdot \mathbf{b} dV = \int_{x^e} \mathbf{N}^T \cdot \mathbf{b} A dx = \int_{-1}^1 \mathbf{N}^T(\xi) \cdot \mathbf{b}(\xi) A |J| d\xi = \sum_{i=1}^n W_i \mathbf{N}^T(\xi_i) \cdot \mathbf{b}(\xi_i) A_i |J_i|$$

- Two-Dimensional Case – Rectangular Elements

 - Quadrature rule

$$\int_{-1}^1 \int_{-1}^1 f(\xi, \eta) d\xi d\eta \approx \int_{-1}^1 \sum_{i=1}^n W_i f(\xi_i, \eta) d\eta = \sum_{j=1}^m W_j \sum_{i=1}^n W_i f(\xi_i, \eta_j) = \sum_{j=1}^m \sum_{i=1}^n W_i W_j f(\xi_i, \eta_j)$$

 - Stiffness equation

$$\int_{A^e} \mathbf{B}^T \cdot \mathbf{D} \cdot \mathbf{B} t dA = \int_{-1}^1 \int_{-1}^1 \mathbf{B}^T(\xi, \eta) \cdot \mathbf{D} \cdot \mathbf{B}(\xi, \eta) t |J| d\xi d\eta = \sum_{i=1}^n \sum_{j=1}^m W_i W_j \mathbf{B}^T(\xi_i, \eta_j) \cdot \mathbf{D}_{ij} \cdot \mathbf{B}(\xi_i, \eta_j) t_{ij} |J_{ij}|$$

$$\int_{A^e} \mathbf{N}^T \cdot \mathbf{b} t dA = \int_{-1}^1 \int_{-1}^1 \mathbf{N}^T(\xi, \eta) \cdot \mathbf{b} t |J| d\xi d\eta = \sum_{i=1}^n \sum_{j=1}^m W_i W_j \mathbf{N}^T(\xi_i, \eta_j) \cdot \mathbf{b}_{ij} t_{ij} |J_{ij}|$$

$$\int_{S^e} \mathbf{N}^T \cdot \mathbf{T} t dS = \int_{-1}^1 \mathbf{N}^T(\xi, \eta_p) \cdot \mathbf{T} t |K| d\xi = \sum_{i=1}^n W_i \mathbf{N}^T(\xi_i, \eta_p) \cdot \mathbf{T}_{ji} t_{ij} |K_{ij}|$$

- Two-Dimensional Case – Triangular Elements

$$\int_{A^e} \mathbf{B}^T \cdot \mathbf{D} \cdot \mathbf{B} t dA = \int_0^{1-\alpha_1} \int_0^{\alpha_1} \mathbf{B}^T(\alpha_1, \alpha_2) \cdot \mathbf{D} \cdot \mathbf{B}(\alpha_1, \alpha_2) t |J| d\alpha_2 d\alpha_1 = \frac{1}{2} \sum_{i=1}^n W_i \mathbf{B}^T(\alpha_1^i, \alpha_2^i) \cdot \mathbf{D}_{ij} \cdot \mathbf{B}(\alpha_1^i, \alpha_2^i) t_i |J_i|$$

8.2. Reduced Integration

- **Q8 element**

$$u = a_0 + a_1\xi + a_2\eta + a_3\xi^2 + a_4\xi\eta + a_5\eta^2 + a_6\xi^2\eta + a_7\xi\eta^2$$

$$v = b_0 + b_1\xi + b_2\eta + b_3\xi^2 + b_4\xi\eta + b_5\eta^2 + b_6\xi^2\eta + b_7\xi\eta^2$$

$$\frac{\partial u}{\partial \xi} = a_1 + 2a_3\xi + a_4\eta + 2a_6\xi\eta + a_7\eta^2 , \quad \frac{\partial v}{\partial \xi} = b_1 + 2b_3\xi + b_4\eta + 2b_6\xi\eta + b_7\eta^2$$

$$\frac{\partial u}{\partial \eta} = a_2 + a_4\xi + 2a_5\eta + a_6\xi^2 + 2a_7\xi\eta , \quad \frac{\partial v}{\partial \eta} = b_2 + b_4\xi + 2b_5\eta + b_6\xi^2 + 2b_7\xi\eta$$

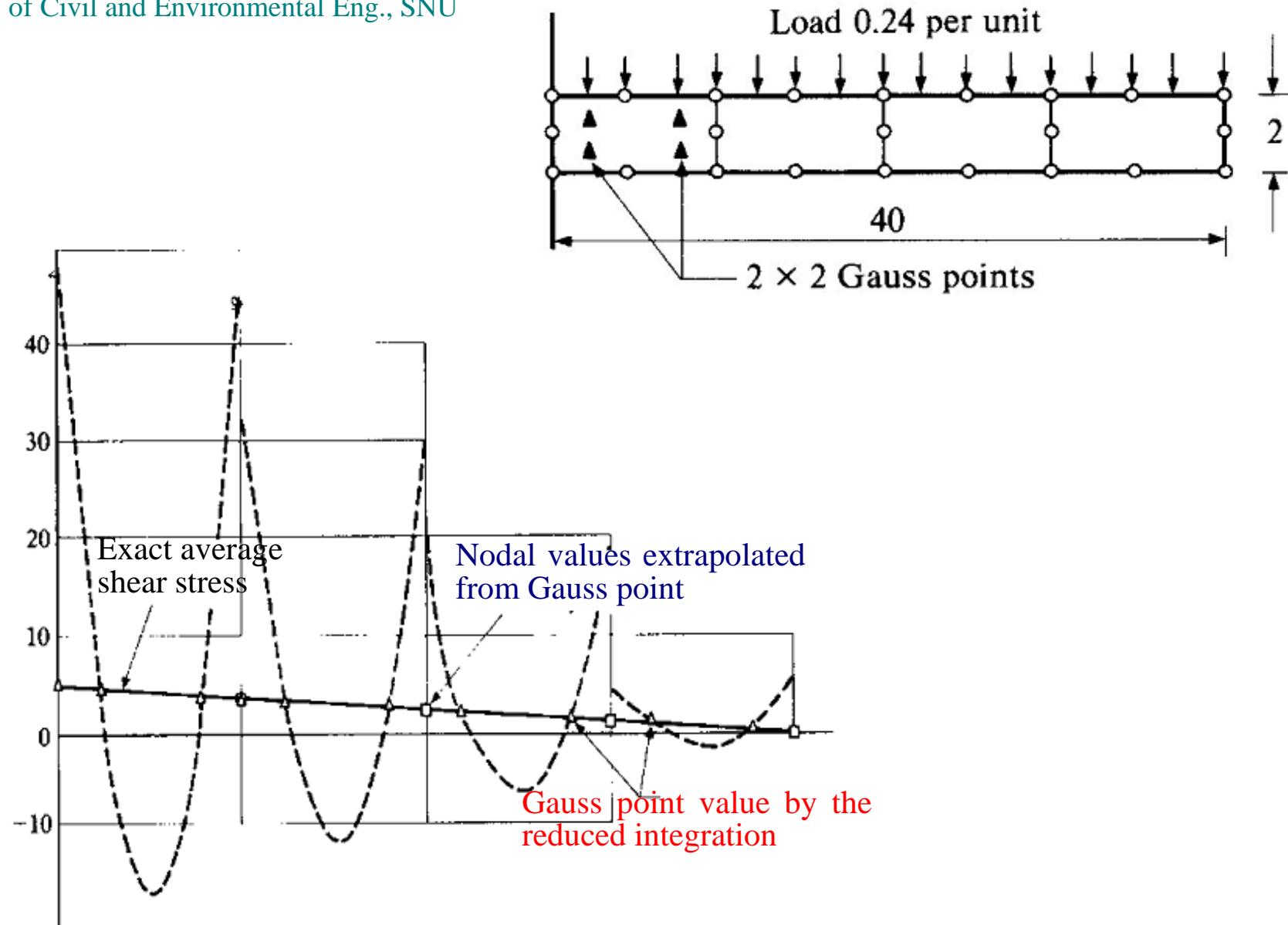
- **Terms in stiffness matrix**

- From complete polynomials: $1, \xi, \eta, \xi^2, \xi\eta, \eta^2$

- From parasitic terms: $\xi\eta, \xi^2\eta, \xi\eta^2, \eta^3, \xi^3, \xi^2\eta^2, \xi^3\eta, \xi\eta^3, \eta^4, \xi^4$

- **Reduced Integration**

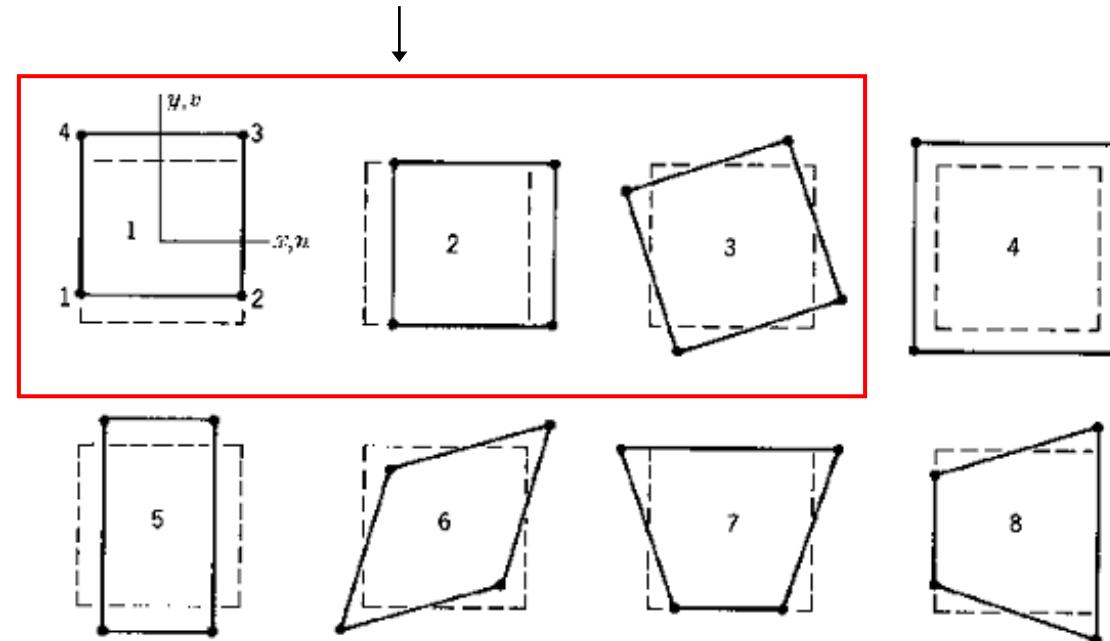
Reduce the integration order by one to eliminate the effect of parasitic terms in the stiffness matrix.



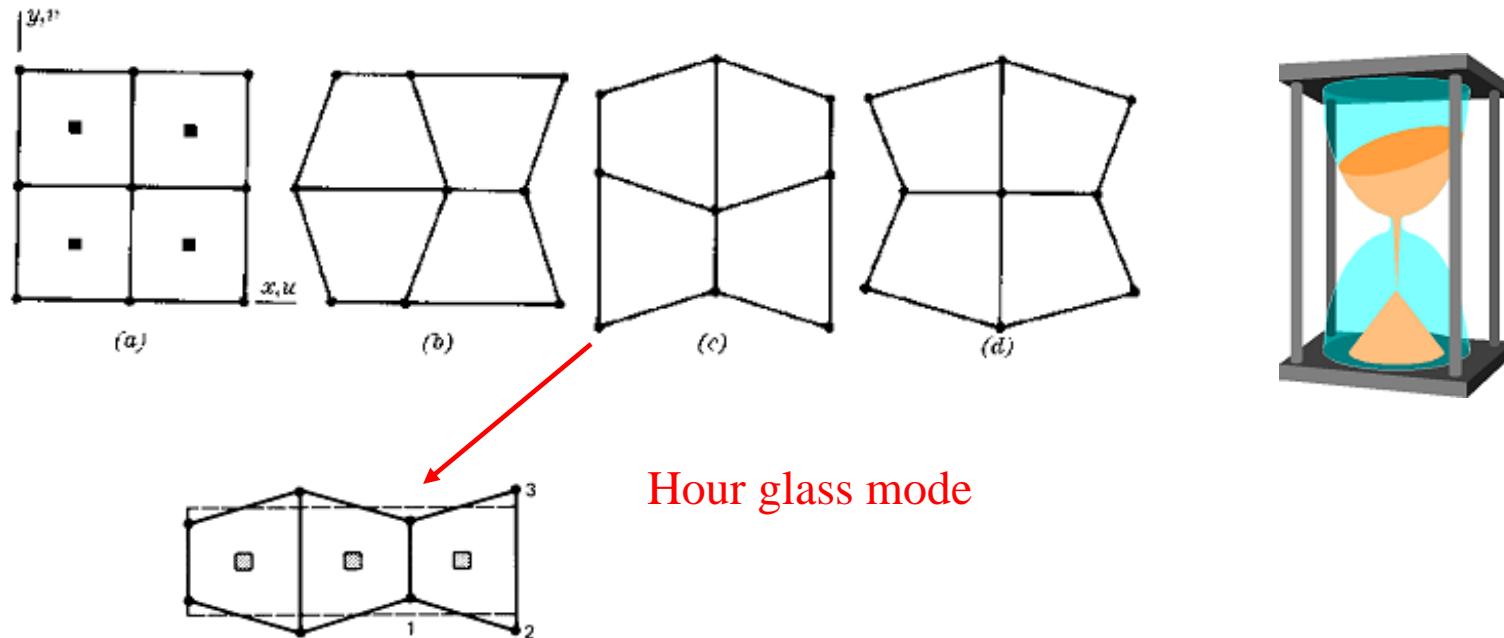
8.3. Spurious Zero Energy mode

- Independent displacement modes of a bilinear element

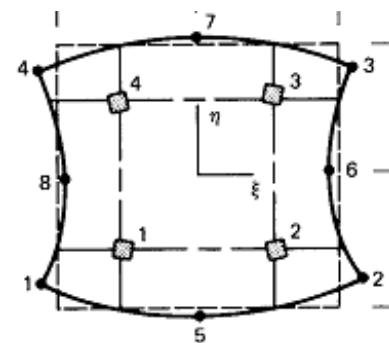
Rigid Body motion – zero energy mode



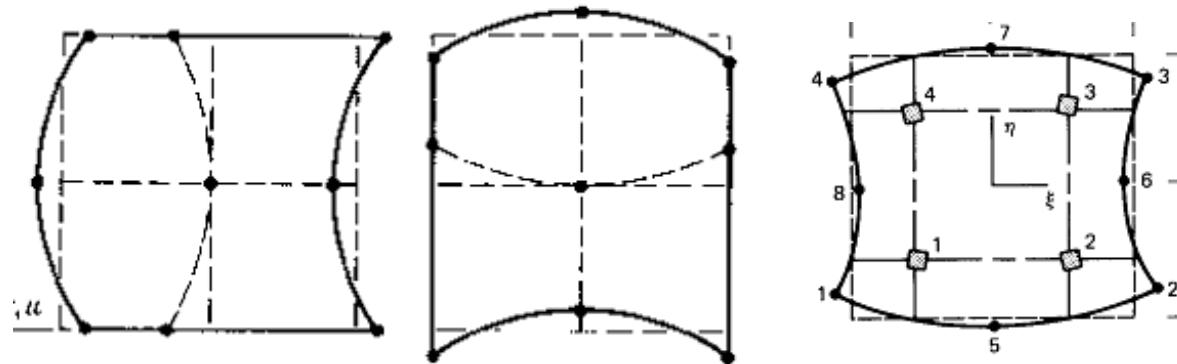
- Spurious zero energy mode



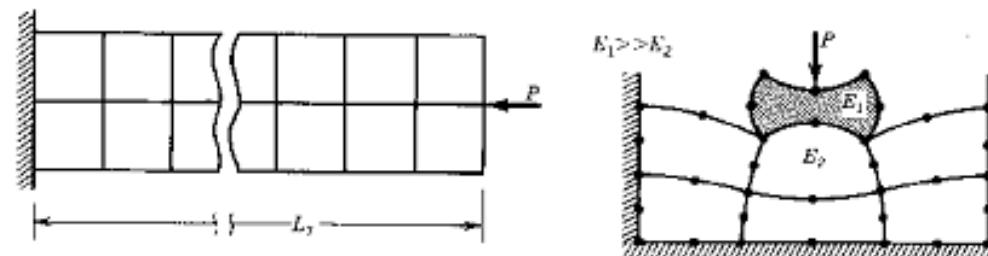
- Zero energy mode of Q8 element



- Zero energy modes of Q9 element



- Near zero energy modes



8.4. Selective Integration

$$\mathbf{D} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 0 & 0 & 0 \\ \frac{\nu}{1-\nu} & 1 & \frac{\nu}{1-\nu} & 0 & 0 & 0 \\ \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} \end{bmatrix}$$

$$\begin{aligned}
 &= \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 0 & 0 & 0 \\ \frac{\nu}{1-\nu} & 1 & \frac{\nu}{1-\nu} & 0 & 0 & 0 \\ \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \frac{E}{2(1+\nu)} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\
 &= \mathbf{D}_N + \mathbf{D}_S
 \end{aligned}$$

$$\int_{V^w} \mathbf{B}^T \mathbf{DB} dV = \int_{V^w} \mathbf{B}^T (\mathbf{D}_N + \mathbf{D}_S) \mathbf{B} dV = \boxed{\int_{V^w} \mathbf{B}^T \mathbf{D}_N \mathbf{B} dV} + \boxed{\int_{V^w} \mathbf{B}^T \mathbf{D}_S \mathbf{B} dV}$$

Full Integration Reduced Integration

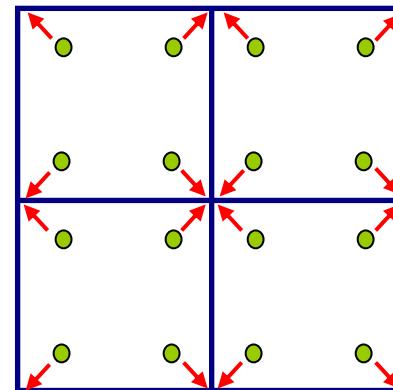
Homework 8

Chapter 9

Convergence Criteria in the Isoparametric Element

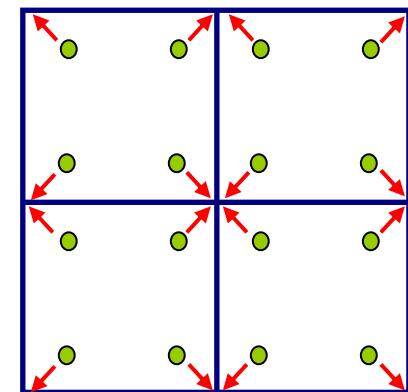
Chapter 10

Miscellaneous Topics



10.1. Stress Evaluation, Smoothing and Loubignac Iteration

- Stress evaluation
 - Stress components should be evaluated at the GP's in each element, not at nodes.
 - The stress field is not uniquely determined on inter-element boundaries.
- Stress smoothing at nodes
 - Continuous stress field can be obtained by extrapolating stresses at the GP's to nodes, and averaging them.
 - The bilinear shape function or the Q9 shape function may be utilized for extrapolation of stress to nodes depending on the integration schemes.
 - Mid-side nodes may be treated as either independent nodes or dependent nodes for the stress field



- Loubignac iteration

$$\sum_e \int_{V^e} \mathbf{B}^T \mathbf{D} \mathbf{B} dV \mathbf{U}^e = \sum_e \int_{V^e} \mathbf{N}^T \mathbf{b} dV + \sum_e \int_{\Gamma_t^e} \mathbf{N}^T \bar{\mathbf{T}} d\Gamma = \mathbf{F}$$

$$\sum_e \int_{V^e} \mathbf{B}^T \mathbf{D} \mathbf{B} dV \mathbf{U}^e = \sum_e \int_{V^e} \mathbf{B}^T \boldsymbol{\sigma} dV = \sum_e \int_{V^e} \mathbf{B}^T (\tilde{\boldsymbol{\sigma}} + \Delta \boldsymbol{\sigma}) dV = \sum_e \int_{V^e} \mathbf{B}^T \tilde{\boldsymbol{\sigma}} dV + \sum_e \int_{V^e} \mathbf{B}^T \mathbf{D} \mathbf{B} dV \Delta \mathbf{U}^e$$

$$\sum_e \int_{V^e} \mathbf{B}^T \mathbf{D} \mathbf{B} dV \Delta \mathbf{U}^e = \mathbf{F} - \sum_e \int_{V^e} \mathbf{B}^T \tilde{\boldsymbol{\sigma}} dV \rightarrow \mathbf{K} \Delta \mathbf{U}^e = \Delta \mathbf{F}$$

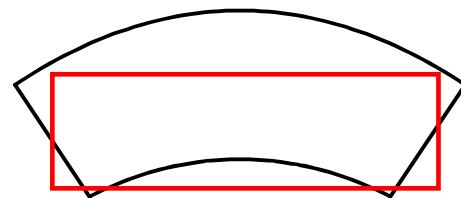
where $\tilde{\boldsymbol{\sigma}}$ denotes the extrapolated and averaged stress field.

10.2. Incompatible Element - Q6

- Deformed shape of Bilinear Element for pure bending



- Correct deformed shape in pure bending



- Behaviors of Bilinear Element for Pure Bending

- Displacement field

$$u = \bar{u} \frac{1}{4}(1-\xi)(1-\eta) - \bar{u} \frac{1}{4}(1+\xi)(1-\eta) + \bar{u} \frac{1}{4}(1+\xi)(1+\eta) - \bar{u} \frac{1}{4}(1-\xi)(1+\eta) = \bar{u} \xi \eta$$

$$v = 0$$

- Strain & Stress field

$$\varepsilon_x = \frac{\partial u}{\partial x} = \eta \frac{\bar{u}}{a} = \frac{\bar{u}y}{ab}, \quad \varepsilon_y = \frac{\partial v}{\partial y} = 0, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\bar{u}x}{ab}$$

$$\sigma_x = \frac{E}{1-\nu^2} \frac{\bar{u}y}{ab}, \quad \sigma_y = \frac{\nu E}{1-\nu^2} \frac{\bar{u}y}{ab}, \quad \sigma_{xy} = \frac{E}{2(1+\nu)} \frac{\bar{u}x}{ab}$$

- Strain Energy - Full Integration

$$\begin{aligned} \Pi_b^f &= \frac{1}{2} \int_{-a-b}^a \int_{-b}^b (\varepsilon_x \sigma_x + \varepsilon_y \sigma_y + \gamma_{xy} \sigma_{xy}) dy dx = \frac{1}{2} \int_{-a-b}^a \int_{-b}^b \left(\frac{E}{1-\nu^2} \left(\frac{\bar{u}y}{ab} \right)^2 + \frac{E}{2(1+\nu)} \left(\frac{\bar{u}x}{ab} \right)^2 \right) dy dx \\ &= \frac{1}{3} \frac{E}{1-\nu^2} \left(2 \frac{b}{a} + (1-\nu) \frac{a}{b} \right) \bar{u}^2 \end{aligned}$$

- Strain Energy - Selective Integration

$$\Pi_b^s = \frac{1}{2} \int_{-a-b}^a \int_{-b}^b \left(\frac{E}{1-\nu^2} \left(\frac{\bar{u}y}{ab} \right)^2 + \frac{E}{2(1+\nu)} \left(\frac{\bar{u}x}{ab} \right)^2 \right) dy dx = \frac{2}{3} \frac{E}{1-\nu^2} \frac{b}{a} \bar{u}^2$$

- Real Behaviors for Pure Bending

 - Strain & Stress field

$$\sigma_x = ky, \sigma_y = 0, \sigma_{xy} = 0 \rightarrow \varepsilon_x = \frac{k}{E}y, \varepsilon_y = -\frac{kv}{E}y, \gamma_{xy} = 0$$

 - Displacement field

$$u = \frac{k}{E}xy + f(y), v = -\frac{1}{2}\frac{kv}{E}y^2 + g(x), \gamma_{xy} = \frac{k}{E}x + f'(y) + g'(x) = 0$$

$$g'(x) + \frac{k}{E}x = -f'(y) = C \rightarrow g(x) = -\frac{1}{2}\frac{k}{E}x^2 + Cx + C_1, f(y) = -Cy + C_2$$

$$u = \frac{k}{E}xy - Cy + C_2, v = -\frac{1}{2}\frac{kv}{E}y^2 - \frac{1}{2}\frac{k}{E}x^2 + Cx + C_1$$

$$x=0 \rightarrow u = -Cy + C_2 = 0 \rightarrow C = C_2 = 0$$

$$x=a, y=b \rightarrow u = \bar{u} \rightarrow k = \frac{E\bar{u}}{ab}, C_1 = \text{Arbitrary}$$

$$u = \frac{\bar{u}}{ab}xy$$

$$v = -\frac{1}{2}\frac{\bar{u}v}{ab}y^2 - \frac{1}{2}\frac{\bar{u}}{ab}x^2 + C_1 = \left(1 - \left(\frac{x}{a}\right)^2\right)\frac{a\bar{u}}{2b} + \left(1 - \left(\frac{y}{b}\right)^2\right)\frac{b\bar{u}}{2a} \quad (\text{in case } v(a,b)=0)$$

- Strain Energy- Exact Solution

$$\Pi_E = \frac{1}{2} \int_{-a-b}^a \int_{-b}^b (\varepsilon_x \sigma_x + \varepsilon_y \sigma_y + \gamma_{xy} \sigma_{xy}) dy dx = \frac{1}{2} \int_{-a-b}^a \int_{-b}^b E \left(\frac{\bar{u}y}{ab} \right)^2 dy dx = \frac{2}{3} E \frac{b}{a} \bar{u}^2$$

- Ratio of strain Energy for $\nu = 0.3$

$$\frac{\Pi_b^f}{\Pi_E} = \frac{1}{1-\nu^2} \left(1 + \frac{1-\nu}{2} \left(\frac{a}{b} \right)^2 \right) \approx 1.1 \left(1 + 0.35 \left(\frac{a}{b} \right)^2 \right), \quad \frac{\Pi_b^s}{\Pi_E} = \frac{1}{1-\nu^2} \approx 1.1$$

The effect of parasitic shear becomes disastrous as the aspect ratio of bilinear element is large.

• Q6 Incompatible Element

- Shape function

$$u = \sum_i N_i^B u_i + a_1(1 - \xi^2) + a_2(1 - \eta^2)$$

$$v = \sum_i N_i^B v_i + a_3(1 - \xi^2) + a_4(1 - \eta^2)$$

a_1, a_2, a_3, a_4 : **nodeless degrees of freedom**

- Element Stiffness Equation

$$\begin{bmatrix} K_{bb} & K_{bi} \\ K_{ib} & K_{ii} \end{bmatrix} \begin{pmatrix} (u) \\ (a) \end{pmatrix} = \begin{pmatrix} (f)^b \\ (f)^i \end{pmatrix}$$

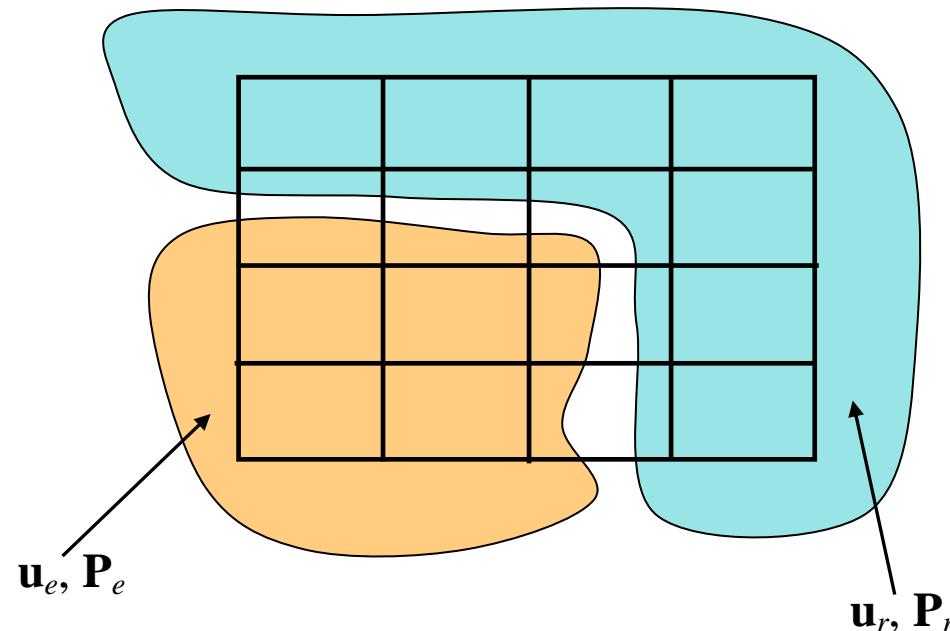
- Static Condensation

$$[K_{ib}](u) + [K_{ii}](a) = (f)^i \rightarrow (a) = [K_{ii}]^{-1}((f)^i - [K_{ib}](u))$$

$$([K_{bb}] - [K_{bi}][K_{ii}]^{-1}[K_{ib}])(u) = (f)^b - [K_{ii}]^{-1}(f)^i \rightarrow [K^{Q6}](u) = (f^{Q6})$$

10.3. Static Condensation & Substructuring

- **Static Condensation** - *Eliminate some DOF prior to a main analysis.*



$$\begin{bmatrix} \mathbf{K}_{ee} & \mathbf{K}_{er} \\ \mathbf{K}_{re} & \mathbf{K}_{rr} \end{bmatrix} \begin{pmatrix} \mathbf{u}_e \\ \mathbf{u}_r \end{pmatrix} = \begin{pmatrix} \mathbf{P}_e \\ \mathbf{P}_r \end{pmatrix}$$

$$\underline{\mathbf{K}_{er}\mathbf{u}_r + \mathbf{K}_{ee}\mathbf{u}_e = \mathbf{P}_e} \rightarrow \mathbf{u}_e = (\mathbf{K}_{ee})^{-1}(\mathbf{P}_e - \mathbf{K}_{er}\mathbf{u}_r)$$

$$\mathbf{K}_{rr}\mathbf{u}_r + \mathbf{K}_{re}\mathbf{u}_e = \mathbf{P}_r$$

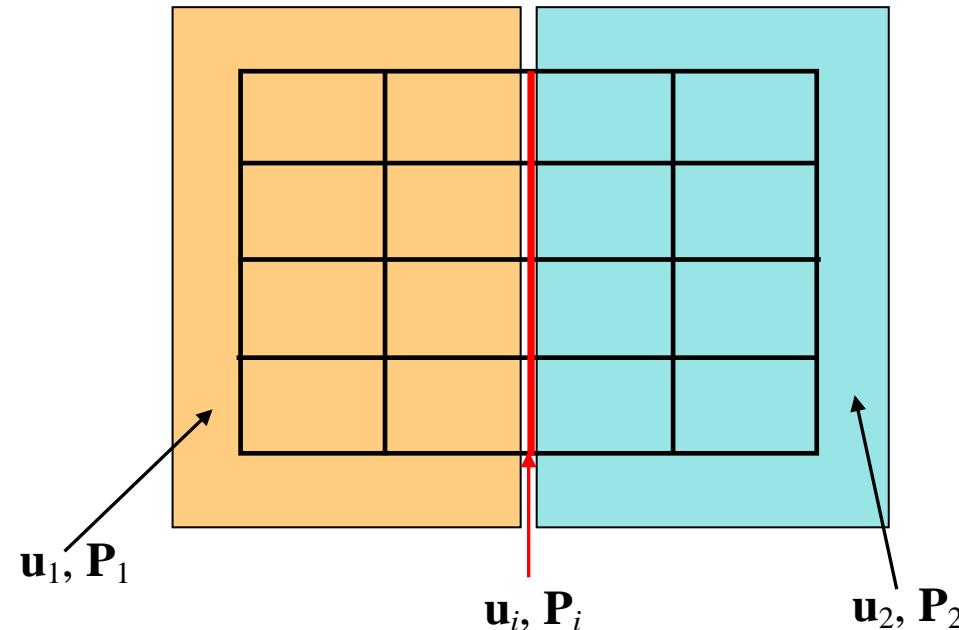
$$\mathbf{K}_{rr}\mathbf{u}_r + \mathbf{K}_{re}(\mathbf{K}_{ee})^{-1}(\mathbf{P}_e - \mathbf{K}_{er}\mathbf{u}_r) = \mathbf{P}_r$$

$$\underline{(\mathbf{K}_{rr} - \mathbf{K}_{re}(\mathbf{K}_{ee})^{-1}\mathbf{K}_{er})\mathbf{u}_r = \mathbf{P}_r - \mathbf{K}_{re}(\mathbf{K}_{ee})^{-1}\mathbf{P}_e}$$

From Gauss elimination point of view

$$\begin{bmatrix} \mathbf{K}_{ee} & \mathbf{K}_{er} \\ 0 & \mathbf{K}_{rr} - \mathbf{K}_{re}(\mathbf{K}_{ee})^{-1}\mathbf{K}_{er} \end{bmatrix} \begin{pmatrix} \mathbf{u}_e \\ \mathbf{u}_r \end{pmatrix} = \begin{pmatrix} \mathbf{P}_e \\ \mathbf{P}_r - \mathbf{K}_{re}(\mathbf{K}_{ee})^{-1}\mathbf{P}_e \end{pmatrix}$$

- **Substructuring**



- Substructure 1

$$\begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{1i} \\ \mathbf{K}_{il} & \mathbf{K}_{ii}^1 \end{bmatrix} \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_i \end{pmatrix} = \begin{pmatrix} \mathbf{P}_1 \\ \mathbf{P}_i^1 \end{pmatrix}$$

$$\underline{\underline{(\mathbf{K}_{ii}^1 - \mathbf{K}_{i1}(\mathbf{K}_{11})^{-1}\mathbf{K}_{1i})\mathbf{u}_i = \mathbf{P}_i^1 - \mathbf{K}_{i1}(\mathbf{K}_{11})^{-1}\mathbf{P}_1}}}$$

- Substructure 2

$$\begin{bmatrix} \mathbf{K}_{22} & \mathbf{K}_{2i} \\ \mathbf{K}_{i2} & \mathbf{K}_{ii}^2 \end{bmatrix} \begin{pmatrix} \mathbf{u}_2 \\ \mathbf{u}_i \end{pmatrix} = \begin{pmatrix} \mathbf{P}_2 \\ \mathbf{P}_i^2 \end{pmatrix}$$

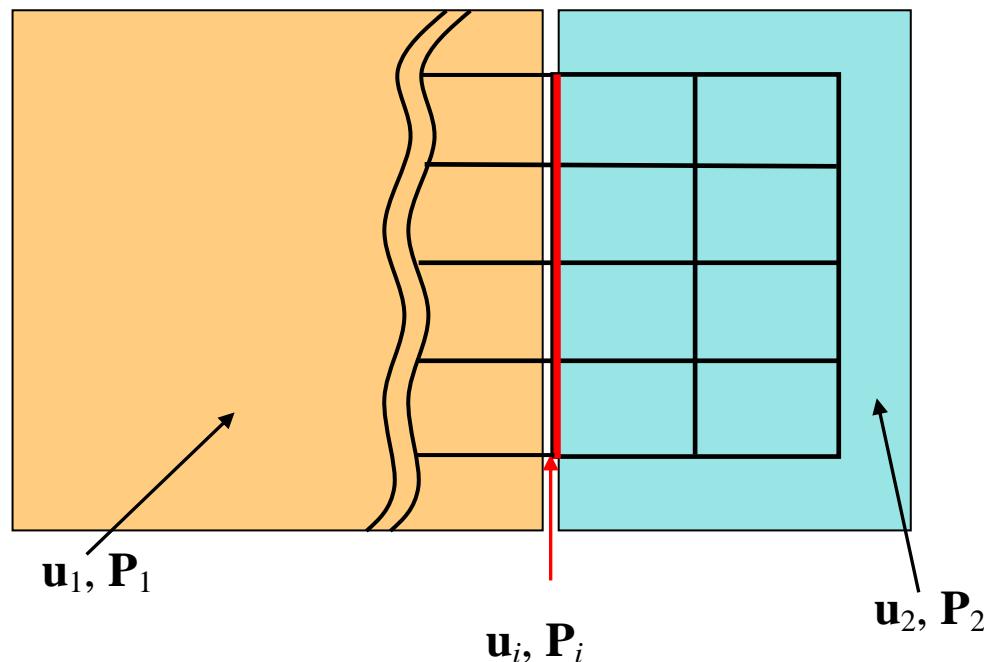
$$\underline{\underline{(\mathbf{K}_{ii}^2 - \mathbf{K}_{i2}(\mathbf{K}_{22})^{-1}\mathbf{K}_{2i})\mathbf{u}_i = \mathbf{P}_i^2 - \mathbf{K}_{i2}(\mathbf{K}_{22})^{-1}\mathbf{P}_2}}$$

- Assembling

$$(\mathbf{K}_{ii}^1 + \mathbf{K}_{ii}^2 - \mathbf{K}_{i1}(\mathbf{K}_{11})^{-1}\mathbf{K}_{1i} - \mathbf{K}_{i2}(\mathbf{K}_{22})^{-1}\mathbf{K}_{2i})\mathbf{u}_i = \mathbf{P}_i^1 + \mathbf{P}_i^2 - \mathbf{K}_{i1}(\mathbf{K}_{11})^{-1}\mathbf{P}_1 - \mathbf{K}_{i2}(\mathbf{K}_{22})^{-1}\mathbf{P}_2$$

$$(\mathbf{K}_{ii}^1 + \mathbf{K}_{ii}^2 - \mathbf{K}_{i1}(\mathbf{K}_{11})^{-1}\mathbf{K}_{1i} - \mathbf{K}_{i2}(\mathbf{K}_{22})^{-1}\mathbf{K}_{2i})\mathbf{u}_i = \mathbf{P}_i - \mathbf{K}_{i1}(\mathbf{K}_{11})^{-1}\mathbf{P}_1 - \mathbf{K}_{i2}(\mathbf{K}_{22})^{-1}\mathbf{P}_2$$

- Partial Substructuring



- Substructure 1

$$\begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{1i} \\ \mathbf{K}_{i1} & \mathbf{K}_{ii}^1 \end{bmatrix} \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_i \end{pmatrix} = \begin{pmatrix} \mathbf{P}_1 \\ \mathbf{P}_i^1 \end{pmatrix}$$

$$\underline{(\mathbf{K}_{ii}^1 - \mathbf{K}_{i1}(\mathbf{K}_{11})^{-1}\mathbf{K}_{1i})\mathbf{u}_i = \mathbf{P}_i^1 - \mathbf{K}_{i1}(\mathbf{K}_{11})^{-1}\mathbf{P}_1}$$

- Substructure 2

$$\begin{bmatrix} \mathbf{K}_{22} & \mathbf{K}_{2i} \\ \mathbf{K}_{i2} & \mathbf{K}_{ii}^2 \end{bmatrix} \begin{pmatrix} \mathbf{u}_2 \\ \mathbf{u}_i \end{pmatrix} = \begin{pmatrix} \mathbf{P}_2 \\ \mathbf{P}_i^2 \end{pmatrix}$$

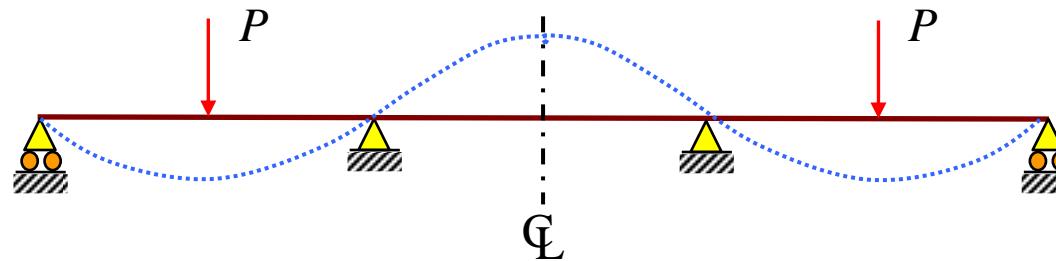
- Assembling

$$\begin{bmatrix} \mathbf{K}_{22} & \mathbf{K}_{2i} \\ \mathbf{K}_{i2} & \mathbf{K}_{ii}^2 + \mathbf{K}_{ii}^1 - \mathbf{K}_{i1}(\mathbf{K}_{11})^{-1}\mathbf{K}_{1i} \end{bmatrix} \begin{pmatrix} \mathbf{u}_2 \\ \mathbf{u}_i \end{pmatrix} = \begin{pmatrix} \mathbf{P}_2 \\ \boxed{\mathbf{P}_i^1 + \mathbf{P}_i^2} - \mathbf{K}_{i1}(\mathbf{K}_{11})^{-1}\mathbf{P}_1 \end{pmatrix}$$

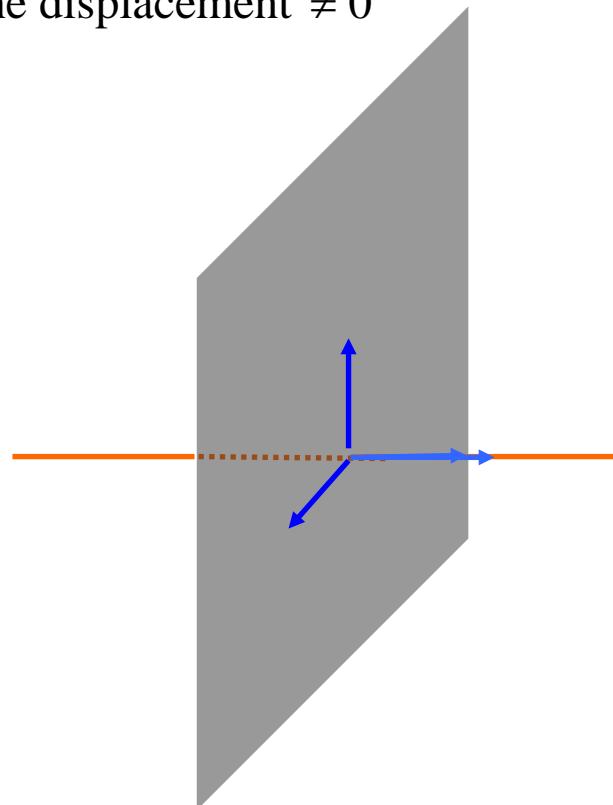
↓

$$\mathbf{P}_i$$

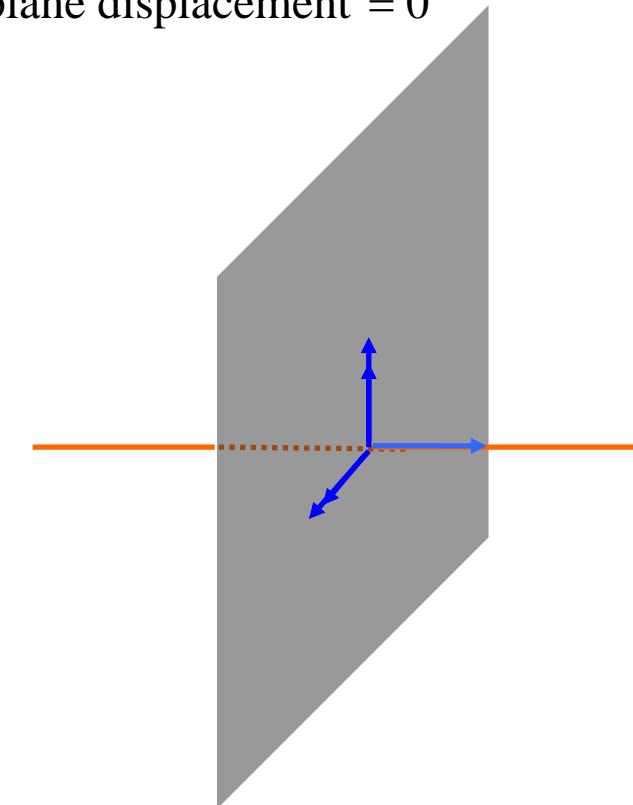
10.4 Symmetry of Structure



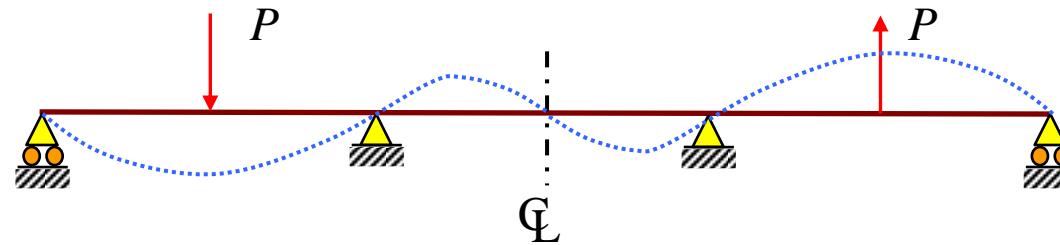
In plane displacement $\neq 0$



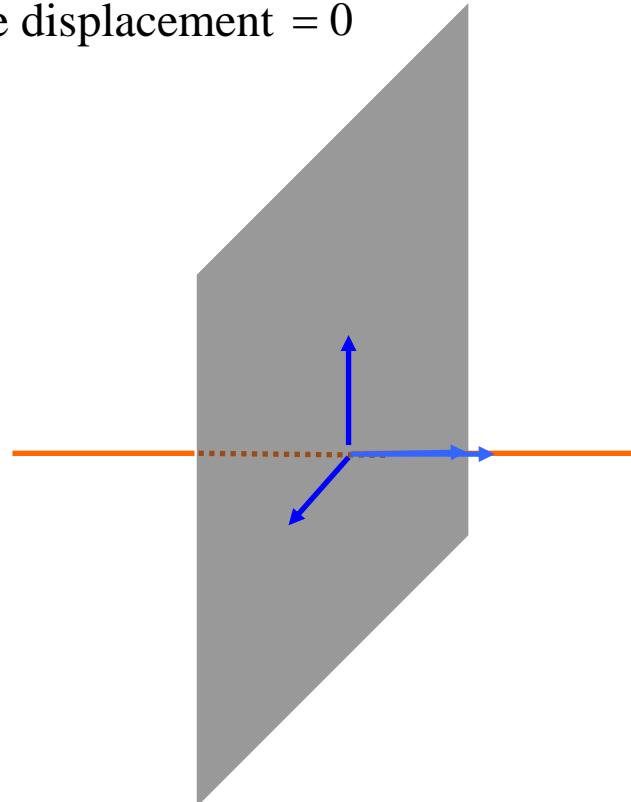
Out of plane displacement = 0



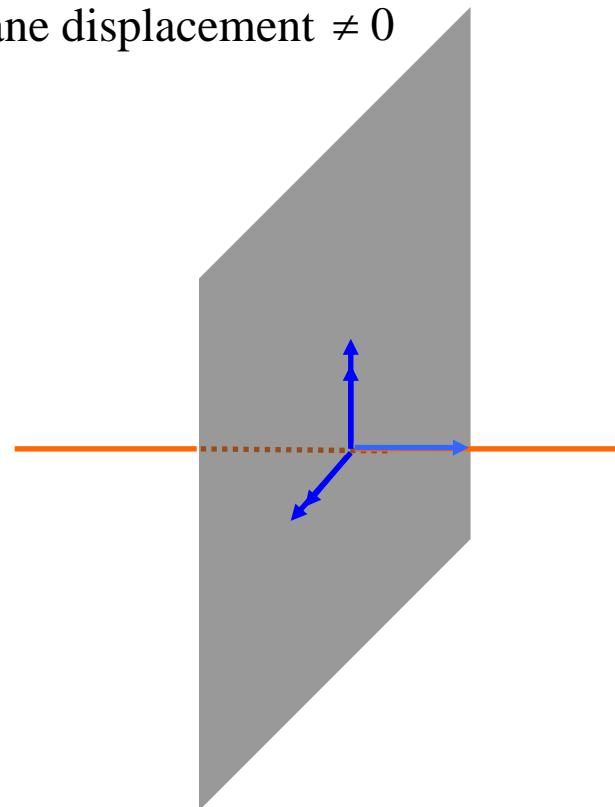
Anti-Symmetry



In plane displacement = 0

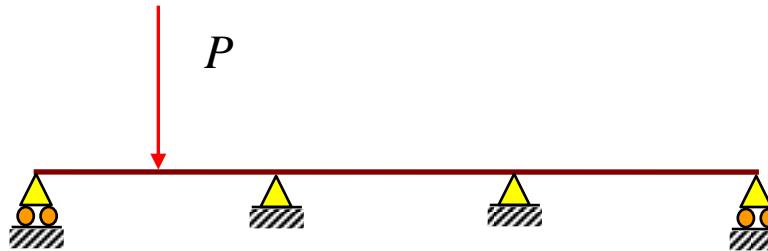


Out of plane displacement $\neq 0$

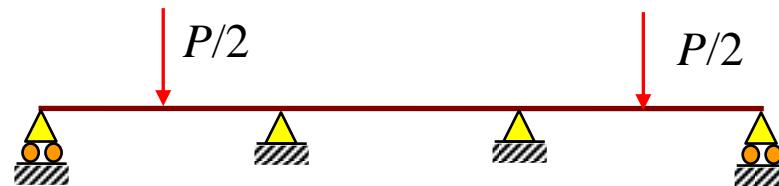


Non-Symmetric loading

- General loading



- Decomposition



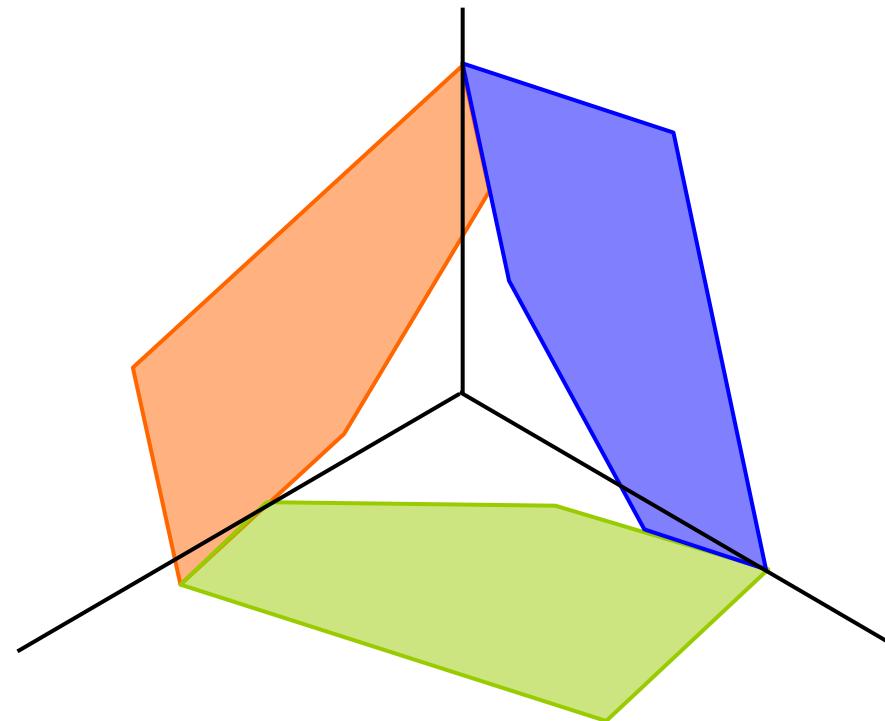
Symmetric Loading



Anti-Symmetric Loading

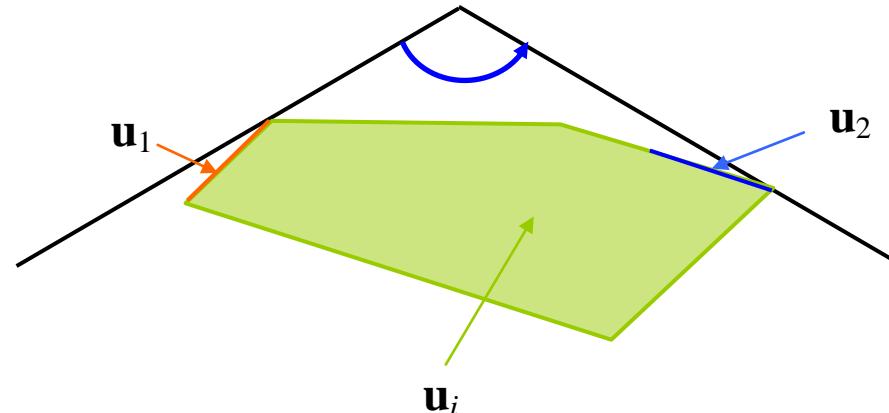
$$\begin{pmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{P}_S \\ \mathbf{P}_S \end{pmatrix} + \begin{pmatrix} \mathbf{P}_A \\ -\mathbf{P}_A \end{pmatrix} \rightarrow \begin{cases} \mathbf{P}_S = \frac{\mathbf{P}_1 + \mathbf{P}_2}{2} \\ \mathbf{P}_A = \frac{\mathbf{P}_1 - \mathbf{P}_2}{2} \end{cases}$$

Cyclic Symmetry



- Structural Resistance force in a segment

$$\begin{pmatrix} \mathbf{F}_1 \\ \mathbf{F}_i \\ \mathbf{F}_2 \end{pmatrix} = \begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{1i} & \mathbf{K}_{12} \\ \mathbf{K}_{i1} & \mathbf{K}_{ii} & \mathbf{K}_{i2} \\ \mathbf{K}_{21} & \mathbf{K}_{2i} & \mathbf{K}_{22} \end{bmatrix} \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_i \\ \mathbf{u}_2 \end{pmatrix}$$



- Compatibility

$$\mathbf{u}_1 = \Gamma \mathbf{u}_2$$

$$\begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_i \\ \mathbf{u}_2 \end{pmatrix} = \begin{bmatrix} \mathbf{0} & \Gamma \\ \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{pmatrix} \mathbf{u}_i \\ \mathbf{u}_2 \end{pmatrix}$$

- Equilibrium

$$\begin{aligned} \mathbf{P}_i &= \mathbf{F}_i \\ \mathbf{P}_2 &= \mathbf{F}_2 + \Gamma^T \mathbf{F}_1 \rightarrow \begin{pmatrix} \mathbf{P}_i \\ \mathbf{P}_2 \end{pmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \Gamma^T & \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{pmatrix} \mathbf{F}_1 \\ \mathbf{F}_i \\ \mathbf{F}_2 \end{pmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \Gamma^T & \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{1i} & \mathbf{K}_{12} \\ \mathbf{K}_{i1} & \mathbf{K}_{ii} & \mathbf{K}_{i2} \\ \mathbf{K}_{21} & \mathbf{K}_{2i} & \mathbf{K}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{0} & \Gamma \\ \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{pmatrix} \mathbf{u}_i \\ \mathbf{u}_2 \end{pmatrix} \end{aligned}$$

- Final Equation

$$\begin{pmatrix} \mathbf{P}_i \\ \mathbf{P}_2 \end{pmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \Gamma^T & \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{K}_{1i} & \mathbf{K}_{11}\Gamma + \mathbf{K}_{12} \\ \mathbf{K}_{ii} & \mathbf{K}_{i1}\Gamma + \mathbf{K}_{i2} \\ \mathbf{K}_{2i} & \mathbf{K}_{21}\Gamma + \mathbf{K}_{22} \end{bmatrix} \begin{pmatrix} \mathbf{u}_i \\ \mathbf{u}_2 \end{pmatrix}$$
$$= \begin{bmatrix} \mathbf{K}_{ii} & \mathbf{K}_{i1}\Gamma + \mathbf{K}_{i2} \\ \Gamma^T \mathbf{K}_{1i} + \mathbf{K}_{2i} & \Gamma^T \mathbf{K}_{11}\Gamma + \Gamma^T \mathbf{K}_{12} + \mathbf{K}_{21}\Gamma + \mathbf{K}_{22} \end{bmatrix} \begin{pmatrix} \mathbf{u}_i \\ \mathbf{u}_2 \end{pmatrix}$$

10.5. Constraints in Discrete Problems

- Minimization Problem

$$\underset{\mathbf{U}}{\text{Min}} \Pi = \frac{1}{2} \mathbf{U}^T \mathbf{K} \mathbf{U} - \mathbf{U}^T \mathbf{P} \quad \text{subject to } \mathbf{A}(\mathbf{U}) = \mathbf{0}$$

$$\mathbf{U} = \begin{pmatrix} \mathbf{U}_f \\ \mathbf{U}_r \end{pmatrix}, \quad \mathbf{K} = \begin{bmatrix} \mathbf{K}_{ff} & \mathbf{K}_{fr} \\ \mathbf{K}_{rf} & \mathbf{K}_{rr} \end{bmatrix} \quad \mathbf{P} = \begin{pmatrix} \mathbf{P}_f \\ \mathbf{P}_r \end{pmatrix}$$

- 1st order Optimality condition

$$\frac{\partial \Pi}{\partial \mathbf{U}} = 0 \rightarrow \mathbf{K} \mathbf{U} - \mathbf{P} = 0$$

- Usual homogenous support conditions

$$\mathbf{U}_r = 0$$

$$\underset{\mathbf{U}_f}{\text{Min}} \Pi = \frac{1}{2} \mathbf{U}_f^T \mathbf{K}_f \mathbf{U}_f - \mathbf{U}_f^T \mathbf{P}_f \rightarrow \mathbf{K}_f \mathbf{U}_f = \mathbf{P}_f$$

- Non-homogenous support conditions (support Settlement)

$$\mathbf{U}_r = \bar{\mathbf{U}}_r$$

$$\begin{aligned}\Pi &= \frac{1}{2}(\mathbf{U}_f^T \mathbf{K}_{ff} \mathbf{U}_f + \bar{\mathbf{U}}_r^T \mathbf{K}_{rf} \mathbf{U}_f + \mathbf{U}_f^T \mathbf{K}_{fr} \bar{\mathbf{U}}_r + \bar{\mathbf{U}}_r^T \mathbf{K}_{rr} \bar{\mathbf{U}}_r) - \mathbf{U}_f^T \mathbf{P}_f - \mathbf{U}_r^T \mathbf{P}_r \\ &= \frac{1}{2}(\mathbf{U}_f^T \mathbf{K}_{ff} \mathbf{U}_f + 2\mathbf{U}_f^T \mathbf{K}_{fr} \bar{\mathbf{U}}_r + \bar{\mathbf{U}}_r^T \mathbf{K}_{rr} \bar{\mathbf{U}}_r) - \mathbf{U}_f^T \mathbf{P}_f - \mathbf{U}_r^T \mathbf{P}_r\end{aligned}$$

$$\underset{\mathbf{U}}{\text{Min}} \Pi = \underset{\mathbf{U}_f}{\text{Min}} \Pi \rightarrow \mathbf{K}_{ff} \mathbf{U}_f + \mathbf{K}_{fr} \bar{\mathbf{U}}_r - \mathbf{P}_f = 0$$

$$\mathbf{K}_{ff} \mathbf{U}_f = \mathbf{P}_f - \mathbf{K}_{fr} \bar{\mathbf{U}}_r$$

or

$$\delta \Pi = \frac{\partial \Pi}{\partial \mathbf{U}} \delta \mathbf{U} = \frac{\partial \Pi}{\partial \mathbf{U}_f} \delta \mathbf{U}_f + \frac{\partial \Pi}{\partial \mathbf{U}_r} \delta \mathbf{U}_r = \frac{\partial \Pi}{\partial \mathbf{U}_f} \delta \mathbf{U}_f = 0 \rightarrow \frac{\partial \Pi}{\partial \mathbf{U}_f} = 0$$

- General Non-homogenous Linear Constraints

$$\mathbf{A}(\mathbf{U}) = \mathbf{R}\mathbf{U} - \mathbf{r}_0 = 0 \rightarrow \sum_{j=1}^{ndof} r_{ij} u_j = r_{i0} \quad , i = 1, \dots, ncon$$

$$\mathbf{R}\mathbf{U} = \mathbf{R}_r \mathbf{U}_r + \mathbf{R}_f \mathbf{U}_f = \mathbf{r}_0 \rightarrow \mathbf{U}_r = -\mathbf{R}_r^{-1} \mathbf{R}_f \mathbf{U}_f + \mathbf{R}_r^{-1} \mathbf{r}_0 = \mathbf{C} \mathbf{U}_f + \mathbf{C}_0$$

$$\mathbf{U} = \begin{pmatrix} \mathbf{U}_f \\ \mathbf{U}_r \end{pmatrix} = \begin{pmatrix} \mathbf{I} \\ \mathbf{C} \end{pmatrix} \mathbf{U}_f + \begin{pmatrix} \mathbf{0} \\ \mathbf{C}_0 \end{pmatrix}$$

$$\begin{aligned} \Pi &= \frac{1}{2} \mathbf{U}^T \mathbf{K} \mathbf{U} - \mathbf{U}^T \mathbf{P} \\ &= \frac{1}{2} \mathbf{U}_f^T [\mathbf{K}_{ff} + \mathbf{K}_{fr} \mathbf{C} + \mathbf{C}^T \mathbf{K}_{rf} + \mathbf{C}^T \mathbf{K}_{rr} \mathbf{C}] \mathbf{U}_f + \mathbf{U}_f^T (\mathbf{K}_{fr} \mathbf{C}_0 + \mathbf{C}^T \mathbf{K}_{rr} \mathbf{C}_0) + \\ &\quad \frac{1}{2} \mathbf{C}_0^T \mathbf{K}_{rr} \mathbf{C}_0 - \mathbf{U}_f^T (\mathbf{P}_f + \mathbf{C}^T \mathbf{P}_r) \end{aligned}$$

$$[\mathbf{K}_{ff} + \mathbf{K}_{fr} \mathbf{C} + \mathbf{C}^T \mathbf{K}_{rf} + \mathbf{C}^T \mathbf{K}_{rr} \mathbf{C}] \mathbf{U}_f = (\mathbf{P}_f + \mathbf{C}^T \mathbf{P}_r) - (\mathbf{K}_{fr} \mathbf{C}_0 + \mathbf{C}^T \mathbf{K}_{rr} \mathbf{C}_0)$$

- Lagrange Multiplier

$$\underset{\mathbf{U}, \lambda}{\text{Min}} \bar{\Pi} = \frac{1}{2} \mathbf{U}^T \mathbf{K} \mathbf{U} - \mathbf{U}^T \mathbf{P} + \boldsymbol{\lambda}^T (\mathbf{R} \mathbf{U} - \mathbf{r}_0) \rightarrow \frac{\partial \bar{\Pi}}{\partial \mathbf{U}} = 0, \quad \frac{\partial \bar{\Pi}}{\partial \boldsymbol{\lambda}} = 0$$

$$\left. \begin{aligned} \frac{\partial \bar{\Pi}}{\partial \mathbf{U}} &= \mathbf{K} \mathbf{U} - \mathbf{P} + \boldsymbol{\lambda}^T \frac{\partial \mathbf{A}(\mathbf{U})}{\partial \mathbf{U}} = \mathbf{K} \mathbf{U} - \mathbf{P} + \boldsymbol{\lambda}^T \mathbf{R} = 0 \\ \frac{\partial \bar{\Pi}}{\partial \boldsymbol{\lambda}} &= \mathbf{A}(\mathbf{U}) = \mathbf{R} \mathbf{U} - \mathbf{r}_0 = 0 \end{aligned} \right\} \rightarrow \begin{bmatrix} \mathbf{K} & \mathbf{R}^T \\ \mathbf{R} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{U} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \mathbf{P} \\ \mathbf{r}_0 \end{pmatrix}$$

$$\mathbf{K} \mathbf{U} - \mathbf{P} + \mathbf{R}^T \boldsymbol{\lambda} = 0 \rightarrow \mathbf{U} = \mathbf{K}^{-1}(\mathbf{P} - \mathbf{R}^T \boldsymbol{\lambda})$$

$$\mathbf{R} \mathbf{U} - \mathbf{r}_0 = \mathbf{R} \mathbf{K}^{-1}(\mathbf{P} - \mathbf{R}^T \boldsymbol{\lambda}) - \mathbf{r}_0 = 0 \rightarrow \boldsymbol{\lambda} = (\mathbf{R} \mathbf{K}^{-1} \mathbf{R}^T)^{-1} (\mathbf{R} \mathbf{K}^{-1} \mathbf{P} - \mathbf{r}_0)$$

$$\begin{aligned} \mathbf{U} &= \mathbf{K}^{-1}(\mathbf{P} - \mathbf{R}^T (\mathbf{R} \mathbf{K}^{-1} \mathbf{R}^T)^{-1} (\mathbf{R} \mathbf{K}^{-1} \mathbf{P} - \mathbf{r}_0)) \\ &= (\mathbf{K}^{-1} - \mathbf{K}^{-1} \mathbf{R}^T (\mathbf{R} \mathbf{K}^{-1} \mathbf{R}^T)^{-1} \mathbf{R} \mathbf{K}^{-1}) \mathbf{P} + \mathbf{K}^{-1} \mathbf{R}^T (\mathbf{R} \mathbf{K}^{-1} \mathbf{R}^T)^{-1} \mathbf{r}_0 \end{aligned}$$

10.6. Constraints in Continuous Problems

- **Lagrange Multiplier**

$$\underset{\mathbf{u}}{\text{Min } \Pi} \quad \text{subject to} \quad \mathbf{A}(\mathbf{u}) = \mathbf{0}$$

where Π is the original functional derived from the minimization principle or equivalent, and $\mathbf{A}(\mathbf{u}) = \mathbf{0}$ denotes an additional set of constraints, which may be defined in some volume, on some surface, or at some points. For the simplicity of derivation, only constraints specified along a surface is considered in this note.

$$\mathbf{A}(\mathbf{u}) = \mathbf{Lu} - \mathbf{r}_0 = 0 \quad \text{on } S$$

$$\begin{aligned} \underset{u_i, \lambda}{\text{Min } \bar{\Pi}(u_i, \lambda)} &= \Pi(u) + \int_S \lambda (\mathbf{Lu} - \mathbf{r}_0) dS \\ &= \frac{1}{2} \int_V \varepsilon_{ij}^h \sigma_{ij}^h dV - \int_V u_i^h b_i dV - \int_{S_t} u_i^h \bar{T}_i dS + \int_S \lambda (\mathbf{Lu} - \mathbf{r}_0) dS \end{aligned}$$

By discretizing $\lambda = \bar{\mathbf{N}}\Lambda^e$ in an element and the displacement field in usual way,

$$\begin{aligned}\bar{\Pi}(u_i, \lambda) &\approx \frac{1}{2} \mathbf{U}^T \mathbf{K} \mathbf{U} - \mathbf{U}^T \mathbf{f} + \sum_e (\Lambda^e)^T \int_{S_e} \bar{\mathbf{N}}^T \mathbf{L} \mathbf{N} dS \mathbf{U}^e - \sum_e (\Lambda^e)^T \int_{S_e} \bar{\mathbf{N}}^T \mathbf{r}_0 dS \\ &= \frac{1}{2} \mathbf{U}^T \mathbf{K} \mathbf{U} - \mathbf{U}^T \mathbf{f} + \Lambda^T \int_{S_e} \bar{\mathbf{N}}^T \mathbf{L} \mathbf{N} dS \mathbf{U} - \Lambda^T \int_{S_e} \bar{\mathbf{N}}^T \mathbf{r}_0 dS \\ &= \frac{1}{2} \mathbf{U}^T \mathbf{K} \mathbf{U} - \mathbf{U}^T \mathbf{f} + \Lambda^T \mathbf{R} \mathbf{U} - \Lambda^T \mathbf{q}\end{aligned}$$

Therefore, the stationary condition for modified energy functional becomes

$$\left. \begin{array}{l} \frac{\partial \bar{\Pi}}{\partial \mathbf{U}} = \mathbf{K} \mathbf{U} - \mathbf{f} + \mathbf{R}^T \Lambda = 0 \\ \frac{\partial \bar{\Pi}}{\partial \Lambda} = \mathbf{R} \mathbf{U} - \mathbf{q} = 0 \end{array} \right\} \rightarrow \begin{bmatrix} \mathbf{K} & \mathbf{R}^T \\ \mathbf{R} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{U} \\ \Lambda \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{q} \end{pmatrix}$$

The solution procedure from this point is exactly the same as that of discrete problems. *The last and the most important question is what kind of interpolation function has to be employed for the Lagrange multiplier.*

- **Penalty Function**

$$\underset{u}{\text{Min}} \quad \bar{\Pi}(\mathbf{u}) = \Pi(\mathbf{u}) + \frac{\alpha}{2} \int_S \mathbf{A}^T(\mathbf{u}) \cdot \mathbf{A}(\mathbf{u}) dS$$

$$\begin{aligned} \bar{\Pi}(\mathbf{u}) &\approx \frac{1}{2} \int_V \varepsilon_{ij}^h \sigma_{ij}^h dV - \int_V u_i^h b_i dV - \int_{S_t} u_i^h \bar{T}_i dS + \frac{\alpha}{2} \int_S (\mathbf{L}\mathbf{u} - \mathbf{r}_0)^T (\mathbf{L}\mathbf{u} - \mathbf{r}_0) dS \\ &= \frac{1}{2} \mathbf{U}^T \mathbf{K} \mathbf{U} - \mathbf{U}^T \mathbf{f} + \frac{\alpha}{2} \sum_e \left\{ (\mathbf{U}^e)^T \int_{S_e} (\mathbf{L}\mathbf{N})^T \mathbf{L}\mathbf{N} dS \mathbf{U}^e - 2(\mathbf{U}^e)^T \int_{S_e} (\mathbf{L}\mathbf{N})^T \mathbf{r}_0 dS + \int_{S_e} \mathbf{r}_0^T \mathbf{r}_0 dS \right\} \\ &= \frac{1}{2} \mathbf{U}^T \mathbf{K} \mathbf{U} - \mathbf{U}^T \mathbf{f} + \frac{\alpha}{2} (\mathbf{U}^T \mathbf{K}_R \mathbf{U} - 2\mathbf{U}^T \mathbf{q} + \mathbf{C}) \end{aligned}$$

$$\underset{u}{\text{Min}} \quad \bar{\Pi}(\mathbf{u}) \rightarrow \frac{\partial \bar{\Pi}}{\partial \mathbf{U}} = 0 \rightarrow (\mathbf{K} + \alpha \mathbf{K}_R) \mathbf{U} = \mathbf{f} + \alpha \mathbf{q}$$

Chapter 11

Problems with Higher Continuity Requirement – Beams

$$\nabla^4 = ??$$

11.1 C¹-Formulation

- Governing Equation

$$EI \frac{d^4 w}{dx^4} = q$$

- Weak Form of the governing equation

$$\int_0^l \hat{w} (EI \frac{d^4 w}{dx^4} - q) dx = 0 \text{ for all admissible } \hat{w}$$

- Integration by part twice

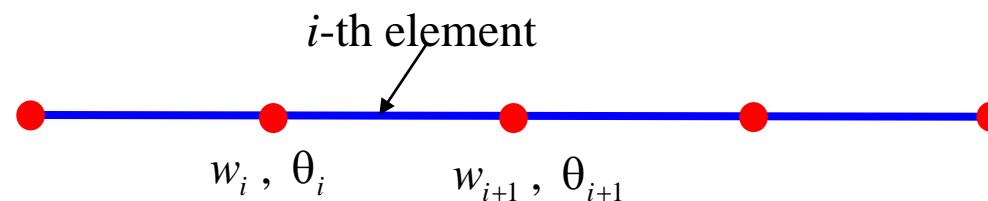
$$\hat{w} EI \frac{d^3 w}{dx^3} \Big|_0^l - \frac{d\hat{w}}{dx} EI \frac{d^2 w}{dx^2} \Big|_0^l + \int_0^l \frac{d^2 \hat{w}}{dx^2} EI \frac{d^2 w}{dx^2} dx - \int_0^l \hat{w} q dx = 0$$

$$\int_0^l \frac{d^2 \hat{w}}{dx^2} EI \frac{d^2 w}{dx^2} dx = \int_0^l \hat{w} q dx + \hat{w} V \Big|_0^l - \hat{\theta} M \Big|_0^l \rightarrow \int_0^l \frac{d^2 \hat{w}}{dx^2} EI \frac{d^2 w}{dx^2} dx = \int_0^l \hat{w} q dx$$

- Continuity Requirement

The second derivative of the displacement field has to be piecewise-continuous for the valid finite element formulation. Therefore, not only w but also $\theta = dw/dx$ should be continuous, which can be achieved by defining nodal degrees of freedom for w and θ and imposing continuity of both DOF at each node. Based on the aforementioned discussions, the displacement field in an element should be defined by displacements and rotational angles at the both ends of an element.

$$\int_0^l \frac{d^2 \hat{w}}{dx^2} EI \frac{d^2 w}{dx^2} dx = \sum_e \int_{l^e}^{l^e} \frac{d^2 \hat{w}}{dx^2} EI \frac{d^2 w}{dx^2} dx$$



- Hermitian Shape Functions

$$w = N_1 w_l + N_2 \theta_l + N_3 w_r + N_4 \theta_r$$

where $w^e(0) = w_i$, $\left. \frac{dw^e}{dx} \right|_{x=0} = \theta_i$, $w^e(l_e) = w_{i+1}$, $\left. \frac{dw^e}{dx} \right|_{x=l_e} = \theta_{i+1}$

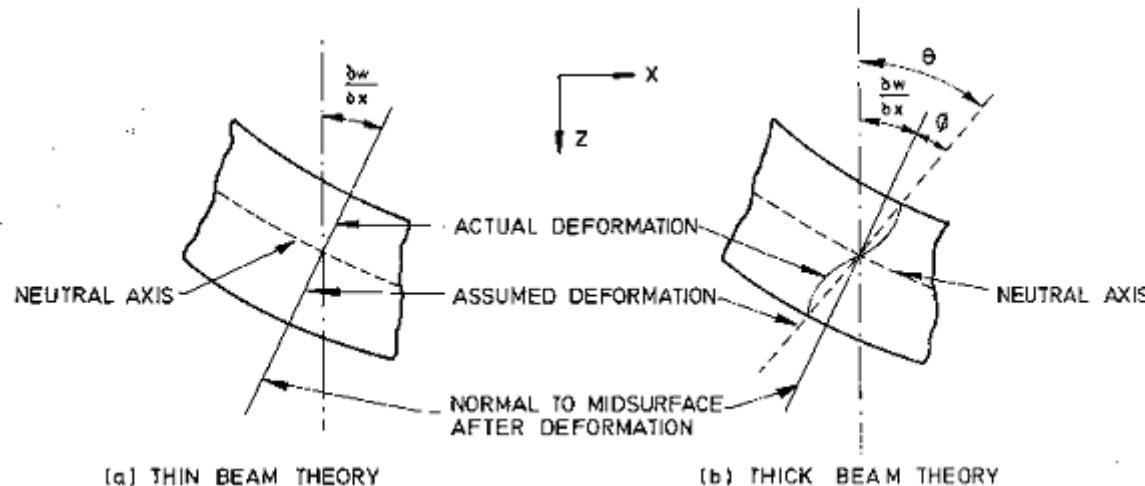
$$N_1 = 1 - 3 \frac{x^2}{l_e^2} + 2 \frac{x^3}{l_e^3}, \quad N_2 = x - 2 \frac{x^2}{l_e} + \frac{x^3}{l_e^2}$$

$$N_3 = 3 \frac{x^2}{l_e^2} - 2 \frac{x^3}{l_e^3}, \quad N_4 = -\frac{x^2}{l_e} + \frac{x^3}{l_e^2}$$

The Hermitian shape functions are the four homogenous solutions of the beam equation.

$$\mathbf{K} = \int_0^l \left(\frac{d^2 \mathbf{N}}{dx^2} \right)^T EI \frac{d^2 \mathbf{N}}{dx^2} dx = \frac{EI_e}{L_e} \begin{bmatrix} \frac{12}{L_e^2} & \frac{6}{L_e} & -\frac{12}{L_e^2} & \frac{6}{L_e} \\ \frac{6}{L_e} & 4 & -\frac{6}{L_e} & 2 \\ -\frac{12}{L_e^2} & -\frac{6}{L_e} & \frac{12}{L_e^2} & -\frac{6}{L_e} \\ \frac{6}{L_e} & 2 & -\frac{6}{L_e} & 4 \end{bmatrix}$$

11.2 C⁰ Formulation



- Displacement fields

w : total deflection

θ : total rotation $\rightarrow u_x = -\theta y$

- Strain fields

$$\varepsilon_x = \frac{du_x}{dx} = -\frac{d\theta}{dx} y, \quad \gamma = \frac{dw}{dx} + \frac{du_x}{dy} = \frac{dw}{dx} - \theta$$

$$\begin{aligned}
 \Pi &= \frac{1}{2} \int_0^l (\sigma_x \varepsilon_x + \tau_{xy} \gamma_{xy}) dV - \int_0^l w q dx \\
 &= \frac{1}{2} \int_0^l \left(\frac{d\theta}{dx} E y^2 \frac{d\theta}{dx} + \left(\frac{dw}{dx} - \theta \right) G \left(\frac{dw}{dx} - \theta \right) \right) dV - \int_0^l w q dx \\
 &= \frac{1}{2} \left(\int_0^l \frac{d\theta}{dx} EI \frac{d\theta}{dx} dx + \int_0^l \left(\frac{dw}{dx} - \theta \right) GA_0 \left(\frac{dw}{dx} - \theta \right) dx \right) - \int_0^l w q dx
 \end{aligned}$$

Therefore the usual C^0 continuity is required for both w and θ .

$$w = \sum_i N_i^w w_i = \mathbf{N}^w \Delta_w^e, \quad \theta = \sum_i N_i^\theta \theta_i = \mathbf{N}^\theta \Delta_\theta^e, \quad \Delta_w^e = (w_1 \quad w_2 \quad \cdots \quad w_n)^T, \quad \Delta_\theta^e = (\theta_1 \quad \theta_2 \quad \cdots \quad \theta_m)^T$$

$$\frac{d\theta}{dx} = \frac{d\mathbf{N}^\theta}{dx} \Delta_\theta^e = \mathbf{B}_\theta \Delta_\theta^e = (\mathbf{0}, \mathbf{B}_\theta) \begin{pmatrix} \Delta_w^e \\ \Delta_\theta^e \end{pmatrix} = \mathbf{B}_M \Delta^e, \quad ,$$

$$\frac{dw}{dx} = \frac{d\mathbf{N}^w}{dx} \Delta_w^e = \mathbf{B}_w \Delta_w^e \rightarrow \frac{dw}{dx} - \theta = \mathbf{B}_w \Delta_w^e - \mathbf{N}^\theta \Delta_\theta^e = (\mathbf{B}_w, -\mathbf{N}^\theta) \begin{pmatrix} \Delta_w^e \\ \Delta_\theta^e \end{pmatrix} = \mathbf{B}_S \Delta^e$$

$$\begin{aligned}
 \Pi &= \frac{1}{2} \sum_e (\Delta^e)^T \left(\int_0^{l_e} \mathbf{B}_M^T EI \mathbf{B}_M dx + \int_0^{l_e} \mathbf{B}_S^T GA_0 \mathbf{B}_S dx \right) \Delta^e - \sum_e (\Delta^e)^T \int_0^{l_e} \mathbf{N} q dx \\
 &= \frac{1}{2} \sum_e (\Delta^e)^T (\mathbf{K}_M^e + \mathbf{K}_S^e) (\Delta^e) - \sum_e (\Delta^e)^T (\mathbf{f}^e) = \frac{1}{2} (\Delta)^T (\mathbf{K}_M + \mathbf{K}_S) (\Delta) - (\Delta)^T \mathbf{f} \\
 &= \frac{1}{2} \Delta^T \mathbf{K} \Delta - \Delta^T \mathbf{f}
 \end{aligned}$$

$$\underline{\underline{\frac{\partial \Pi}{\partial \Delta} = \mathbf{K} \Delta - {}^T \mathbf{f} = 0}}}$$

- Continuous form

$$\frac{\Pi}{EI} = \frac{1}{2} \left(\int_0^l \frac{d\theta}{dx} \frac{d\theta}{dx} dx + \frac{GA_0 l^2}{EI} \frac{1}{l^2} \int_0^l \left(\frac{dw}{dx} - \theta \right) \left(\frac{dw}{dx} - \theta \right) dx \right) - \int_0^l \frac{wq}{EI} dx$$

$$\lambda = \frac{GA_0 l^2}{EI} = \frac{6\kappa}{(1+\nu)} \frac{l^2}{t^2} \text{ for rectangular beam.}$$

As $\lambda \rightarrow \infty$ $\frac{1}{l^2} \int_0^l \left(\frac{dw}{dx} - \theta \right) \left(\frac{dw}{dx} - \theta \right) dx \rightarrow 0$ to keep the total potential energy finite.

- Difficulty in finite element method

$$\frac{\Pi l_e}{EI} = \frac{1}{2} \left(\sum_e \int_0^1 \frac{d\theta}{d\xi} \frac{d\theta}{d\xi} d\xi + \frac{GA_0 l^2}{EI} \frac{l_e^2}{l^2} \int_0^1 \left(\frac{1}{l^e} \frac{dw}{d\xi} - \theta \right) \left(\frac{1}{l^e} \frac{dw}{d\xi} - \theta \right) d\xi \right) - l_e^2 \sum_e \int_0^1 \frac{wq}{EI} d\xi$$

In case $\theta \neq dw/d\xi$, the ratio $\frac{l_e^2}{l^2}$ should be the order of $1/\lambda$ to keep total potential energy finite,

which means the length of an element should be approximately the same order of the height of a beam.

11.3 Timoshenko Beam

- Governing Equation

$$EI \frac{d^2\theta}{dx^2} + GA_0 \left(\frac{dw}{dx} - \theta \right) = 0 , \quad GA_0 \left(\frac{d^2w}{dx^2} - \frac{d\theta}{dx} \right) = -p$$

- Weak Form

$$\int_0^l \delta\theta \left(EI \frac{d^2\theta}{dx^2} + GA_0 \left(\frac{dw}{dx} - \theta \right) \right) dx = \delta\theta EI \frac{d\theta}{dx} \Big|_0^l - \int_0^l \frac{d\delta\theta}{dx} EI \frac{d\theta}{dx} dx + \int_0^l \delta\theta GA_0 \left(\frac{dw}{dx} - \theta \right) dx = 0$$

$$\int_0^l \delta w \left(GA_0 \left(\frac{d^2w}{dx^2} - \frac{d\theta}{dx} \right) + p \right) dx = \delta w GA_0 \left(\frac{dw}{dx} - \theta \right) \Big|_0^l - \int_0^l \frac{d\delta w}{dx} GA_0 \left(\frac{dw}{dx} - \theta \right) dx + \int_0^l \delta w p dx = 0$$

$$\int_0^l \frac{d\delta\theta}{dx} EI \frac{d\theta}{dx} dx + \int_0^l \left(\frac{d\delta w}{dx} - \delta\theta \right) GA_0 \left(\frac{dw}{dx} - \theta \right) dx = \delta\theta EI \frac{d\theta}{dx} \Big|_0^l + \delta w GA_0 \left(\frac{dw}{dx} - \theta \right) \Big|_0^l + \int_0^l \delta w p dx$$

$$\int_0^l \frac{d\delta\theta}{dx} EI \frac{d\theta}{dx} dx + \int_0^l \left(\frac{d\delta w}{dx} - \delta\theta \right) GA_0 \left(\frac{dw}{dx} - \theta \right) dx = \int_0^l \delta w p dx$$

$$\forall \delta\theta \in \mathbf{v}_\theta \text{ & } \forall \delta w \in \mathbf{v}_w$$

- Boundary Conditions

$$\theta = 0 \text{ or } EI \frac{d\theta}{dx} = 0 \text{ or } w = 0 \text{ or } GA_0 \left(\frac{dw}{dx} - \theta \right) = 0$$

- Elimination of the displacement – Beginning of Nightmare

Differentiation of the first equation and substitution of the second equation into the first equation

$$EI \frac{d^3\theta}{dx^3} + p = 0 \quad (\text{Oh, My God !!!})$$

Unfortunately, we have an odd order differential equation, which does not have the minimum characteristics, and thus is very difficult to solve. At this point, we have to consider the Petrov-Galerkin method seriously !!!

Chapter 12

Mixed Formulation



What is the mixed formulation???

Stress or strain fields are treated and interpolated independently!!!

- Governing Equations and Boundary Conditions

Equilibrium Equation : $\nabla \cdot \boldsymbol{\sigma} + \mathbf{b} = 0$ in V

Constitutive Law : $\boldsymbol{\sigma} = \mathbf{D} : \boldsymbol{\varepsilon}$ in V

Strain-Displacement Relationship : $\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T)$ in V

Displacement Boundary condition : $\mathbf{u} - \bar{\mathbf{u}} = 0$ on S_u

Traction Boundary Condition : $\mathbf{T} - \bar{\mathbf{T}} = 0$ on S_t

Cauchy's Relation on the Boundary : $\mathbf{T} = \boldsymbol{\sigma} \cdot \mathbf{n}$ on S

- Weak statement.

$$\int_V \hat{u}_i (\sigma_{ij,j} + b_i) dV + \int_V \hat{\varepsilon}_{ij} (\varepsilon_{ij} - \frac{1}{2}(u_{i,j} + u_{j,i})) dV - \int_{\Gamma_t} \hat{u}_i (T_i - \bar{T}_i) d\Gamma = 0 \quad \forall \hat{u}_i \in \mathcal{V}_u \quad \hat{\varepsilon}_i \in \mathcal{V}_\varepsilon$$

$$\int_V \hat{u}_i (\sigma_{ij,j} + b_i) dV + \int_V \hat{\varepsilon}_{ij} D_{ijkl} (\varepsilon_{kl} - \frac{1}{2}(u_{k,l} + u_{l,k})) dV - \int_{\Gamma_t} \hat{u}_i (T_i - \bar{T}_i) d\Gamma = 0 \quad \forall \hat{u}_i \in \mathcal{V}_u \quad \hat{\varepsilon}_i \in \mathcal{V}_\varepsilon$$

$$\int_V \frac{\partial \hat{u}_i}{\partial x_j} \sigma_{ij} dV - \int_V \hat{u}_i b_i dV - \int_{\Gamma_t} \hat{u}_i \bar{T}_i d\Gamma + \int_V \hat{\varepsilon}_{ij} D_{ijkl} (\varepsilon_{kl} - \frac{1}{2}(u_{k,l} + u_{l,k})) dV = 0 \quad \forall \hat{u}_i \in \mathcal{V}_u \quad \hat{\varepsilon}_i \in \mathcal{V}_\varepsilon$$

$$\int_V \frac{\partial \hat{u}_i}{\partial x_j} D_{ijkl} \varepsilon_{kl} dV - \int_V \hat{u}_i b_i dV - \int_{\Gamma_t} \hat{u}_i \bar{T}_i d\Gamma + \int_V \hat{\varepsilon}_{ij} D_{ijkl} (\varepsilon_{kl} - \frac{1}{2}(u_{k,l} + u_{l,k})) dV = 0 \quad \forall \hat{u}_i \in \mathcal{V}_u \quad \hat{\varepsilon}_i \in \mathcal{V}_\varepsilon$$

- Interpolations

$$\mathbf{u}^e = \mathbf{N}^e \mathbf{U}^e \quad , \quad \boldsymbol{\varepsilon}^e = \mathbf{N}^e \mathbf{E}^e$$

- Finite element discretization

$$\sum_e (\hat{\mathbf{U}}^e)^T \left(\int_{V^e} \mathbf{B}^T \mathbf{D} \mathbf{N} dV \mathbf{E}^e - \int_V \mathbf{N}^T \mathbf{b} dV - \int_{\Gamma_t} \mathbf{N}^T \bar{\mathbf{T}} d\Gamma \right) + \sum_e (\hat{\mathbf{E}}^e)^T \left(\int_{V^e} \mathbf{N}^T \mathbf{D} \mathbf{N} dV \mathbf{E}^e - \int_V \mathbf{N}^T \mathbf{D} \mathbf{B} dV \mathbf{U}^e \right) = 0$$

for all admissible $\hat{\mathbf{U}}^e$ and $\hat{\mathbf{E}}^e$

$$\mathbf{Q}^T \mathbf{E} = \mathbf{F} , \quad \mathbf{M} \mathbf{E} - \mathbf{Q} \mathbf{U} = 0 \rightarrow \begin{bmatrix} \mathbf{0} & -\mathbf{Q}^T \\ -\mathbf{Q} & \mathbf{M} \end{bmatrix} \begin{pmatrix} \mathbf{U} \\ \mathbf{E} \end{pmatrix} = \begin{pmatrix} -\mathbf{F} \\ \mathbf{0} \end{pmatrix} \rightarrow \mathbf{Q}^T \mathbf{M}^{-1} \mathbf{Q} \mathbf{U} = \mathbf{F}$$

- **Important Question**

What are the admissible function spaces for the displacement field and the strain field?? Can we choose the interpolation shape functions for the displacement and the strain independently ??

Unfortunately, the answer is “No”. In case we choose the function spaces arbitrarily, the solutions of the mixed formulation become very unstable, which is caused by so called “function space interlocking”. The Babuzuka-Brezzi condition (BB condition) states the required relationship between the individual function spaces. This issue is out of scope for this class.

You should be very careful when you use FEM based on the mixed formulation !!!

- Possible choices of function spaces

$$1. \hat{u}_i \in H^1, \hat{\varepsilon}_i \in H^1$$

$$2. \hat{u}_i \in H^1, \hat{\varepsilon}_i \in L_2$$

$$3. \hat{u}_i \in L_2, \hat{\varepsilon}_i \in L_2$$

Which one do you like ? Can you give a proper explanation for your choice ?