이 문서는 서울대학교 공과대학 건설환경공학과 이해성 교수가 1993 ~ 2022 년 까지 약 30년 간 강의한 구조해석 2 강좌를 위하여 제작한 강의노트입니다. 이 강의노트는 비상업적 교육용으로 누구나 사용하고 배포할 수 있으나, 상업적 목 적으로는 사용하기 위하여는 저작자의 사전 동의를 받아야 합니다. 또한 이 노 트의 내용을 인용할 경우에는 적절한 방법을 통하여 출처를 반드시 명기하여야 합니다.

Class Note for Structural Analysis 2

(The last class at SNU, 1993 ~2022)

Spring Semester, 2022



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- Structural Analysis Lab.

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Chapter 1

Slope Deflection Method



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1.0 Comparison of Flexibility Method and Stiffness Method

Flexibility Method

• Remove redundancy (Equilibrium)

• Compatibility

$$\begin{split} \delta_1 &= \delta_2 \\ \frac{X}{k_1} &= \frac{P-X}{k_2} \rightarrow X = \frac{k_1}{k_1 + k_2} P \end{split}$$

Stiffness Method

$$P$$

 k_1
 k_2
• Compatibility

• Equilibrium

$$k_1\delta + k_2\delta = P \rightarrow \delta = \frac{P}{k_1 + k_2}$$

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Department of Civil & Environmental Eng., SNU Flexibility Method

- 1. Release all redundancies.
- 2. Calculate displacements induced by external loads at the released redundancies.
- 3. Apply unit loads and calculate displacements at the released redundancies.
- 4. Construct the flexibility equation by superposing the displacement based on the compatibility conditions.
- 5. Solve the flexibility equation.
- 6. Calculate reactions and other quantities as needed.

- 1. Fix all Degrees of Freedom.
- 2. Calculate fixed end forces induced by external loads at the fixed DOF.
- 3. Apply unit displacements and calculate member end forces at the DOFs.
- 4. Construct the stiffness equation by superposing the member end forces based on the equilibrium equations.
- 5. Solve the stiffness equation.
- 6. Calculate reactions and other quantities as needed.

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1.1 Analysis of Fundamental System

1.1.1 End Rotation

- Flexibility Method
- i) $\theta_B = 0$

$$M_{A} \qquad M_{B}$$

$$M_{B}$$

$$M_{B}$$

$$M_{B}$$

$$M_{B}$$

$$M_{B}$$

$$M_{B}$$

$$M_{B}$$

$$M_{A}$$

$$M_{B}$$

$$M_{A}$$

$$M_{B}$$

$$M_{A}$$

$$M_{B}$$

$$M_{A}$$

$$M_{B}$$

$$M_{B$$

ii) $\theta_A = 0$

$$M_A = -\frac{2EI}{L}\Theta_B, \ M_B = \frac{4EI}{L}\Theta_B$$

• Sign Convention for M :Counterclockwise "+"

• $\theta_A \neq 0$, $\theta_B \neq 0$

$$M_{A} = \frac{4EI}{L} \theta_{A} + \frac{2EI}{L} \theta_{B}$$
$$M_{B} = \frac{2EI}{L} \theta_{A} + \frac{4EI}{L} \theta_{B}$$

1.1.2 Relative motion of joints

• Flexibility Method

or in the new sign convention : $M_A = \frac{6EI}{L}\frac{\Delta}{L}$, $M_B = \frac{6EI}{L}\frac{\Delta}{L}$

• Final Slope-Deflection Equation

$$M_{A} = \frac{4EI}{L} \theta_{A} + \frac{2EI}{L} \theta_{B} + \frac{6EI}{L} \frac{\Delta}{L}$$
$$M_{B} = \frac{2EI}{L} \theta_{A} + \frac{4EI}{L} \theta_{B} + \frac{6EI}{L} \frac{\Delta}{L}$$

• In Case an One End is Hinged

$$M_{A} = \frac{4EI}{L} \theta_{A} + \frac{2EI}{L} \theta_{B} + \frac{6EI}{L} \frac{\Delta}{L} = 0 \rightarrow \frac{2EI}{L} \theta_{A} = -\frac{EI}{L} \theta_{B} - \frac{3EI}{L} \frac{\Delta}{L}$$
$$M_{B} = \frac{2EI}{L} \theta_{A} + \frac{4EI}{L} \theta_{B} + \frac{6EI}{L} \frac{\Delta}{L} = \frac{3EI}{L} \theta_{B} + \frac{3EI}{L} \frac{\Delta}{L}$$

1.1.3 Fixed End Force

•

Both Ends Fixed

 $\frac{qL^2}{12}$ $\frac{qL}{12}$ $\frac{qL}{12}$ $\frac{qL}{12}$ $\frac{qL}{12}$ $\frac{qL}{12}$ $\frac{qL}{12}$ $\frac{qL}{12}$ $\frac{pL}{12}$ $\frac{qL}{12}$ $\frac{$

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• One End Hinged

• Ex.: Uniform load case with a hinged left end

$$M_B^f = -\frac{qL^2}{12} - \frac{qL^2}{24} = -\frac{3qL^2}{24} = -\frac{qL^2}{8} , \quad M_A^f = 0$$

1.1.4 Joint Equilibrium

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1.2 Analysis of Beams

1.2.1 A Fixed-fixed End Beam

• **DOF**: θ_B , Δ_B

• Analysis

i) All fixed : No fixed end forces

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• Construct the Stiffness Equation

$$\begin{split} \sum M_{B}^{i} &= 0 \rightarrow M_{BA}^{1} + M_{BC}^{1} + M_{BC}^{2} + M_{BC}^{2} = 0 \rightarrow 4EI(\frac{1}{a} + \frac{1}{b})\theta_{B} + 6EI(\frac{1}{a^{2}} - \frac{1}{b^{2}})\Delta_{B} = 0\\ \sum V_{B}^{i} &= P \rightarrow V_{BA}^{1} + V_{BC}^{1} + V_{BA}^{2} + V_{BC}^{2} = P \rightarrow \qquad 6EI(\frac{1}{a^{2}} - \frac{1}{b^{2}})\theta_{B} + 12EI(\frac{1}{a^{3}} + \frac{1}{b^{3}})\Delta_{B} = P\\ \theta_{B} &= -\frac{(b-a)a^{2}b^{2}}{2EIl^{3}}P \quad , \quad \Delta_{B} = \frac{a^{3}b^{3}}{3EIl^{3}}P\\ M_{AB} &= M_{AB}^{1} + M_{AB}^{2} = \frac{2EI}{a}\theta_{B} + \frac{6EI}{a^{2}}\Delta_{B} = \frac{ab^{2}}{l^{2}}P ,\\ M_{CB} &= M_{CB}^{1} + M_{CB}^{2} = \frac{2EI}{b}\theta_{B} - \frac{6EI}{b^{2}}\Delta_{B} = -\frac{a^{2}b}{l^{2}}P \end{split}$$

1.2.2 Analysis of a Two-span Continuous Beam (Approach I)

• **DOF**: θ_B , θ_C

• Analysis

i) Fix all DOFs and Calculate FEM.

$$M_{AB}^{f} = \frac{qL^{2}}{12}$$
, $M_{BA}^{f} = -\frac{qL^{2}}{12}$, $M_{BC}^{f} = \frac{qL^{2}}{8}$, $M_{CB}^{f} = -\frac{qL^{2}}{8}$

ii) $\theta_B \neq 0$, $\theta_C = 0$

$$M_{AB}^{1} = \frac{2EI}{L} \Theta_{B}$$
, $M_{BA}^{1} = \frac{4EI}{L} \Theta_{B}$, $M_{BC}^{1} = \frac{8EI}{L} \Theta_{B}$, $M_{CB}^{1} = \frac{4EI}{L} \Theta_{B}$

iii) $\theta_B = 0$, $\theta_C \neq 0$

$$M_{BC}^{2} = \frac{4EI}{L} \theta_{C} \quad , \quad M_{CB}^{2} = \frac{8EI}{L} \theta_{C}$$

• Construct the Stiffness Equation

$$\sum M_{B}^{i} = 0 \to M_{BA}^{f} + M_{BC}^{f} + M_{BA}^{1} + M_{BC}^{1} + M_{BC}^{2} = 0 \to \frac{qL^{2}}{24} + 12\frac{EI}{L}\theta_{B} + 4\frac{EI}{L}\theta_{C} = 0$$

$$\sum M_{C}^{i} = 0 \to M_{CB}^{f} + M_{CB}^{1} + M_{CB}^{2} = 0 \to -\frac{qL^{2}}{8} + 4\frac{EI}{L}\theta_{B} + 8\frac{EI}{L}\theta_{C} = 0$$

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$$\theta_B = -\frac{qL^3}{96EI} \quad , \quad \theta_C = \frac{qL^3}{48EI}$$

• Member End Forces

$$M_{AB} = M_{AB}^{f} + M_{AB}^{1} = \frac{qL^{2}}{12} + \frac{2EI}{L}\theta_{B} = \frac{1}{16}qL^{2}$$

$$M_{BA} = M_{BA}^{f} + M_{BA}^{1} = -\frac{qL^{2}}{12} + \frac{4EI}{L}\theta_{B} = -\frac{1}{8}qL^{2}$$

$$M_{BC} = M_{BC}^{f} + M_{BC}^{1} + M_{BC}^{2} = \frac{qL^{2}}{8} + \frac{8EI}{L}\theta_{B} + \frac{4EI}{L}\theta_{C} = \frac{1}{8}qL^{2}$$

$$M_{CB} = M_{CB}^{f} + M_{CB}^{1} + M_{CB}^{2} = -\frac{qL^{2}}{8} + \frac{4EI}{L}\theta_{B} + \frac{8EI}{L}\theta_{C} = 0$$

• Various Diagram

- Free Body Diagram

- Moment Diagram

1.2.3 Analysis of a Two-span Continuous Beam (Approach II)

- **DOF** : θ_B
- Analysis

i) Fix all DOFs and Calculate FEM.

$$M_{AB}^{f} = \frac{qL^{2}}{12}, \ M_{BA}^{f} = -\frac{qL^{2}}{12}, \ M_{BC}^{f} = \frac{qL^{2}}{8} + \frac{1}{2}\frac{qL^{2}}{8} = \frac{3qL^{2}}{16}$$

ii) $\theta_B \neq 0$

$$M_{AB}^{1} = \frac{2EI}{L} \theta_{B}, \quad M_{BA}^{1} = \frac{4EI}{L} \theta_{B}, \quad M_{BC}^{1} = \frac{6EI}{L} \theta_{B}$$

• Construct Stiffness Equation

$$\sum M_{B} = 0 \to M_{BA}^{f} + M_{BC}^{f} + M_{BA}^{1} + M_{BC}^{1} = 0$$
$$-\frac{qL^{2}}{12} + \frac{3qL^{2}}{16} + 4\frac{EI}{L}\theta_{B} + 6\frac{EI}{L}\theta_{B} = 0 \to \theta_{B} = -\frac{qL^{3}}{96EI}$$

• Member End Forces

$$M_{AB} = M_{AB}^{f} + M_{AB}^{1} = \frac{qL^{2}}{12} + \frac{2EI}{L}\theta_{B} = \frac{1}{16}qL^{2}$$
$$M_{BA} = M_{BA}^{f} + M_{BA}^{1} = -\frac{qL^{2}}{12} + \frac{4EI}{L}\theta_{B} = -\frac{1}{8}qL^{2}$$
$$M_{BC} = M_{BC}^{f} + M_{BC}^{1} = \frac{3qL^{2}}{16} + \frac{6EI}{L}\theta_{B} = \frac{1}{8}qL^{2}$$

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1.2.4 Analysis of a Beam with an Internal Hinge (4 DOFs System)

- **DOF**: θ_B , θ_C^L , θ_C^R , Δ_C
- Analysis
- i) All fixed

$$M_{AB}^{f} = \frac{ql^2}{12}$$
, $M_{BA}^{f} = -\frac{ql^2}{12}$

ii) $\theta_B \neq 0$

$$M_{BC}^{2} = \frac{2EI}{l} \theta_{C}^{L}$$
, $M_{CB}^{2} = \frac{4EI}{l} \theta_{C}^{L}$, $V_{CB}^{2} = \frac{6EI}{l^{2}} \theta_{C}^{L}$

• Construct Stiffness Equation

$$\sum_{i} M_{1}^{i} = 0 \rightarrow -\frac{ql^{2}}{12} + 8\frac{El}{l}\theta_{B} + 2\frac{EI}{l}\theta_{C}^{L} + 6\frac{EI}{l^{2}}\Delta_{C} = 0$$

$$\sum_{i} M_{2}^{i} = 0 \rightarrow 2\frac{EI}{l}\theta_{B} + 4\frac{EI}{l}\theta_{C}^{L} + 6\frac{EI}{l^{2}}\Delta_{C} = 0$$

$$\sum_{i} M_{3}^{i} = 0 \rightarrow 4\frac{EI}{l}\theta_{C}^{R} - 6\frac{EI}{l^{2}}\Delta_{C} = 0$$

$$\sum_{i} V_{4}^{i} = 0 \rightarrow 6\frac{EI}{l^{2}}\theta_{B} + 6\frac{EI}{l^{2}}\theta_{C}^{L} - 6\frac{EI}{l^{2}}\theta_{C}^{R} + 24\frac{EI}{l^{3}}\Delta_{C} = 0$$

- Elimination of θ_C^L and θ_C^R
 - 2nd and 3rd equation

$$2\frac{EI}{l}\theta_{c}^{L} = -(\frac{EI}{l}\theta_{B} + 3\frac{EI}{l^{2}}\Delta_{c}) , 2\frac{EI}{l}\theta_{c}^{R} = 3\frac{EI}{l^{2}}\Delta_{c}$$

- 1st equation

$$-\frac{ql^{2}}{12} + 8\frac{EI}{l}\theta_{B} + 2\frac{EI}{l}\theta_{C}^{L} + 6\frac{EI}{l^{2}}\Delta_{C} =$$

$$-\frac{ql^{2}}{12} + 8\frac{EI}{l}\theta_{B} - (\frac{EI}{l}\theta_{B} + 3\frac{EI}{l^{2}}\Delta_{C}) + 6\frac{EI}{l^{2}}\Delta_{C} = 0 \rightarrow -\frac{ql^{2}}{12} + 7\frac{EI}{l}\theta_{B} + 3\frac{EI}{l^{2}}\Delta_{C} = 0$$

$$-4^{\text{th}} \text{ equation}$$

$$6\frac{EI}{l^{2}}\theta_{B} + 6\frac{EI}{l^{2}}\theta_{C}^{L} - 6\frac{EI}{l^{2}}\theta_{C}^{R} + 24\frac{EI}{l^{3}}\Delta_{C} =$$

$$6\frac{EI}{l^2}\theta_B - 3(\frac{EI}{l^2}\theta_B + 3\frac{EI}{l^3}\Delta_C) - 3(3\frac{EI}{l^3}\Delta_C) + 24\frac{EI}{l^3}\Delta_C = 0 \rightarrow 3\frac{EI}{l^2}\theta_B + 6\frac{EI}{l^3}\Delta_C = 0$$

1.2.5 Analysis of a Beam with an Internal Hinge (2 DOFs System)

- **DOF** : θ_B , Δ_C
- Analysis

i) All fixed

$$M_{AB}^{f} = \frac{ql^2}{12}$$
, $M_{BA}^{f} = -\frac{ql^2}{12}$

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iii) $\Delta_C \neq 0$

• Construct the Stiffness Equation

$$\sum_{i} M_{1}^{i} = 0 \rightarrow -\frac{ql^{2}}{12} + 7\frac{EI}{l}\theta_{B} + 3\frac{EI}{l^{2}}\Delta_{C} = 0$$

$$\sum_{i} V_{2}^{i} = 0 \rightarrow 3\frac{EI}{l^{2}}\theta_{B} + 6\frac{EI}{l^{3}}\Delta_{C} = 0$$

$$\theta_{B} = \frac{ql^{3}}{66EI} , \quad \Delta_{C} = -\frac{ql^{4}}{132EI}$$

$$\frac{2EI}{l}\theta_{C}^{R} = -\frac{3EI}{l}(-\frac{\Delta_{C}}{l}) \rightarrow \theta_{C}^{R} = \frac{3}{2}\frac{\Delta_{C}}{l} = -\frac{ql^{3}}{88EI}$$

$$\theta_{C}^{L} = -\frac{1}{2}\theta_{B} - \frac{3}{2}\frac{\Delta_{C}}{l} = -\frac{ql^{3}}{132EI} + \frac{3}{2}\frac{ql^{3}}{132EI} = \frac{ql^{3}}{264EI}$$

1.2.6 Beam with a Spring Support

• Analysis

i) All fixed

$$M_{AB}^{f} = \frac{ql^2}{12}$$
, $M_{BA}^{f} = -\frac{ql^2}{12}$

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 $k\Delta_C$

• Construct the Stiffness Equation

 $M_{BC}^2 = -M_{DC}^2 = \frac{3EI}{I^2}\Delta_C$

$$\sum_{i} M_{1}^{i} = 0 \rightarrow -\frac{ql^{2}}{12} + 7\frac{EI}{l}\theta_{B} + 3\frac{EI}{l^{2}}\Delta_{C} = 0$$

$$\sum_{i} V_{2}^{i} = 0 \rightarrow \qquad 3\frac{EI}{l^{2}}\theta_{B} + (6\frac{EI}{l^{3}} + k)\Delta_{C} = 0$$

$$\theta_{B} = \frac{1+\alpha}{1+14\alpha/11}\frac{ql^{3}}{66EI} , \quad \Delta_{C} = -\frac{1}{(1+14\alpha/11)}\frac{ql^{4}}{132EI} \quad \text{where} \quad k = \alpha\frac{6EI}{l^{3}}$$

 $V_{BC}^2 = -V_{CB}^2 = -\frac{3EI}{l^2}\Delta_C$, $V_{CD}^2 = -V_{DC}^2 = \frac{3EI}{l^2}\Delta_C$, $V_S^2 = k\Delta_C$

• $\alpha \rightarrow 0$

$$\theta_{\scriptscriptstyle B} = \frac{q l^3}{66 E I} \ , \ \Delta_{\scriptscriptstyle C} = -\frac{q l^4}{132 E I}$$

• $\alpha \rightarrow \infty$

$$\theta_B = \frac{ql^3}{84EI} \quad , \quad \Delta_C = 0$$

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$$M_{BA}^{1} = \frac{4EI}{l} \theta_{B} , \quad M_{BC}^{1} = \frac{3EI}{l} \theta_{B}$$

• Construct the Equilibrium Equation

$$\sum_{i} M_{1}^{i} = 0 \rightarrow 6 \frac{EI}{l} \frac{\delta}{l} - 3 \frac{EI}{l} \frac{\delta}{l} + 4 \frac{EI}{l} \theta_{B} + 3 \frac{EI}{l} \theta_{B} = 0 \rightarrow \theta_{B} = -\frac{3}{7} \frac{\delta}{l}$$

1.2.8 Temperature Change

$$\theta_A = \frac{\alpha (T_2 - T_1)}{2h}l \quad , \quad \theta_B = -\frac{\alpha (T_2 - T_1)}{2h}l$$

• Fixed End Moment

$$M_{A} = \frac{4EI}{L} \theta_{A} + \frac{2EI}{L} \theta_{B} = \frac{\alpha(T_{2} - T_{1})}{h} EI$$
$$M_{B} = \frac{2EI}{L} \theta_{A} + \frac{4EI}{L} \theta_{B} = -\frac{\alpha(T_{2} - T_{1})}{h} EI$$

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1.3 Analysis of Frames

1.3.1 A Portal Frame without Sidesway

- **DOF**: θ_B , θ_C
- Analysis
 - All fixed $M_{BC}^{0} = \frac{Pl}{8}$, $M_{CB}^{0} = -\frac{Pl}{8}$
- ii) $\theta_B \neq 0$

i)

$$M_{AB}^{1} = \frac{2EI_{1}}{l} \theta_{B} , \quad M_{BA}^{1} = \frac{4EI_{1}}{l} \theta_{B}$$
$$M_{BC}^{1} = \frac{4EI_{2}}{l} \theta_{B} , \quad M_{CB}^{1} = \frac{2EI_{2}}{l} \theta_{B}$$

iii) $\theta_C \neq 0$

$$M_{BC}^{2} = \frac{2EI_{2}}{l} \theta_{C} , \quad M_{CB}^{2} = \frac{4EI_{2}}{l} \theta_{C}$$
$$M_{CD}^{2} = \frac{4EI_{1}}{l} \theta_{C} , \quad M_{DC}^{2} = \frac{2EI_{1}}{l} \theta_{C}$$

• Construct the Stiffness Equation

$$\begin{split} \sum M_B^i &= 0 \rightarrow \quad \frac{Pl}{8} + (\frac{4EI_1}{l} + \frac{4EI_2}{l})\theta_B + \frac{2EI_2}{l}\theta_C = 0\\ \sum M_C^i &= 0 \rightarrow -\frac{Pl}{8} + \frac{2EI_2}{l}\theta_B + (\frac{4EI_1}{l} + \frac{4EI_2}{l})\theta_C = 0\\ -\theta_B &= \theta_C = \frac{1}{4EI_1 + 2EI_2}\frac{Pl^2}{8} \end{split}$$

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• Member End Forces

$$\begin{split} M_{AB} &= \frac{2EI_{1}}{l} \theta_{B} = -\frac{2EI_{1}}{4EI_{1} + 2EI_{2}} \frac{Pl}{8} \\ M_{BA} &= \frac{4EI_{1}}{l} \theta_{B} = -\frac{4EI_{1}}{4EI_{1} + 2EI_{2}} \frac{Pl}{8} \\ M_{BC} &= \frac{Pl}{8} + \frac{4EI_{2}}{l} \theta_{B} + \frac{2EI_{2}}{l} \theta_{C} = \frac{4EI_{1}}{4EI_{1} + 2EI_{2}} \frac{Pl}{8} \\ M_{CB} &= -\frac{Pl}{8} + \frac{2EI_{2}}{l} \theta_{B} + \frac{4EI_{2}}{l} \theta_{C} = -\frac{4EI_{1}}{4EI_{1} + 2EI_{2}} \frac{Pl}{8} \\ M_{CD} &= \frac{4EI_{1}}{l} \theta_{C} = \frac{4EI_{1}}{4EI_{1} + 2EI_{2}} \frac{Pl}{8} \\ M_{DC} &= \frac{2EI_{1}}{l} \theta_{C} = \frac{2EI_{1}}{4EI_{1} + 2EI_{2}} \frac{Pl}{8} \end{split}$$

• In case $EI_1 = EI_2$

$$M_{AB} = -\frac{Pl}{24}$$
, $M_{BA} = -\frac{Pl}{12}$, $M_{BC} = \frac{Pl}{12}$, $M_{CB} = -\frac{Pl}{12}$, $M_{CD} = \frac{Pl}{12}$, $M_{DC} = \frac{Pl}{24}$

1.3.2 A Portal Frame without Sidesway – hinged supports

- **DOF**: θ_B , θ_C
- Analysis

i) All fixed

$$M_{BC}^{0} = \frac{Pl}{8}$$
 , $M_{CB}^{0} = -\frac{Pl}{8}$

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ii) $\theta_B \neq 0$ $M_{BA}^1 = \frac{3EI_1}{l} \theta_B$ $M_{BC}^1 = \frac{4EI_2}{l} \theta_B$, $M_{CB}^1 = \frac{2EI_2}{l} \theta_B$

iii)
$$\theta_C \neq 0$$

$$M_{BC}^{2} = \frac{2EI_{2}}{l} \theta_{C} \quad , \quad M_{CB}^{2} = \frac{4EI_{2}}{l} \theta_{C}$$
$$M_{CD}^{2} = \frac{3EI_{1}}{l} \theta_{C}$$

• Construct the Stiffness Equation

$$\sum M_B^i = 0 \rightarrow \frac{Pl}{8} + (\frac{3EI_1}{l} + \frac{4EI_2}{l})\theta_B + \frac{2EI_2}{l}\theta_C = 0$$

$$\sum M_C^i = 0 \rightarrow -\frac{Pl}{8} + \frac{2EI_2}{l}\theta_B + (\frac{3EI_1}{l} + \frac{4EI_2}{l})\theta_C = 0$$

$$-\theta_B = \theta_C = \frac{1}{3EI_1 + 2EI_2}\frac{Pl^2}{8}$$

• Member End Forces

$$\begin{split} M_{AB} &= 0 \\ M_{BA} &= \frac{3EI_1}{l} \theta_B = -\frac{3EI_1}{3EI_1 + 2EI_2} \frac{Pl}{8} \\ M_{BC} &= \frac{Pl}{8} + \frac{4EI_2}{l} \theta_B + \frac{2EI_2}{l} \theta_C = \frac{3EI_1}{3EI_1 + 2EI_2} \frac{Pl}{8} \\ M_{CB} &= -\frac{Pl}{8} + \frac{2EI_2}{l} \theta_B + \frac{4EI_2}{l} \theta_C = -\frac{3EI_1}{3EI_1 + 2EI_2} \frac{Pl}{8} \\ M_{CD} &= \frac{3EI_1}{l} \theta_C = \frac{3EI_1}{3EI_1 + 2EI_2} \frac{Pl}{8} \\ \end{split}$$

• In case of $EI_1 = EI_2$ $M_{AB} = 0$, $M_{BA} = -\frac{3}{40}Pl$, $M_{BC} = \frac{3}{40}Pl$, $M_{CB} = -\frac{3}{40}Pl$, $M_{CD} = \frac{3}{40}Pl$, $M_{DC} = 0$

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1.3.3 A Frame with a horizontal force

- **DOF**: θ_B , Δ
- Analysis
- i) All fixed : None fixed end moment ii) $\theta_B \neq 0$

$$M_{AB}^{1} = \frac{2EI}{l} \theta_{B} , \quad M_{BA}^{1} = \frac{4EI}{l} \theta_{B}$$
$$M_{BC}^{1} = \frac{3EI}{l} \theta_{B} , \quad V_{BA}^{1} = \frac{6EI}{l^{2}} \theta_{B}$$

iii) $\Delta \neq 0$

$$M_{AB}^{2} = \frac{6EI}{l^{2}}\Delta , \quad M_{BA}^{2} = \frac{6EI}{l^{2}}\Delta$$
$$V_{BA}^{2} = \frac{12EI}{l^{3}}\Delta$$

• Construct the stiffness equation

$$\sum M_B^i = 0 \rightarrow (\frac{4EI}{l} + \frac{3EI}{l})\theta_B + \frac{6EI}{l^2}\Delta = 0$$

$$\sum V^i = P \rightarrow \frac{6EI}{l^2}\theta_B + \frac{12EI}{l^3}\Delta = P$$

$$\theta_B = -\frac{Pl^2}{8EI}, \Delta = \frac{7Pl^3}{48EI}$$

• Member end forces

$$M_{AB} = \frac{2EI}{l} \theta_{B} + \frac{6EI}{l^{2}} \Delta = \frac{5}{8} Pl , \quad M_{BA} = \frac{4EI}{l} \theta_{B} + \frac{6EI}{l^{2}} \Delta = \frac{3}{8} Pl , \quad M_{BC} = \frac{3EI}{l} \theta_{B} = -\frac{3}{8} Pl$$

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1.3.4 A Portal Frame with an Unsymmetrical Load

- **DOF**: θ_B , θ_C , Δ
- Analysis
- i) All fixed

$$M^{0}_{BC} = \frac{Pab^{2}}{l^{2}} \ , \ M^{0}_{CB} = -\frac{Pa^{2}b}{l^{2}}$$

ii) $\theta_B \neq 0$

$$M_{AB}^{1} = \frac{2EI}{l} \theta_{B} , \quad M_{BA}^{1} = \frac{4EI}{l} \theta_{B}$$
$$M_{BC}^{1} = \frac{4EI}{l} \theta_{B} , \quad M_{CB}^{1} = \frac{2EI}{l} \theta_{B} , \quad V_{BA}^{1} = \frac{6EI}{l^{2}} \theta_{B}$$

iii) $\theta_C \neq 0$

$$M_{BC}^{2} = \frac{2EI}{l} \theta_{C} , \quad M_{CB}^{2} = \frac{4EI}{l} \theta_{C}$$
$$M_{CD}^{2} = \frac{4EI}{l} \theta_{C} , \quad M_{DC}^{2} = \frac{2EI}{l} \theta_{C} , \quad V_{CD}^{2} = \frac{6EI}{l^{2}} \theta_{C}$$

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iv) $\Delta \neq 0$

$$M_{AB}^{3} = M_{BA}^{3} = \frac{6EI}{l^{2}}\Delta , \quad M_{CD}^{3} = M_{DC}^{3} = \frac{6EI}{l^{2}}\Delta ,$$
$$V_{BA}^{3} = V_{CD}^{3} = \frac{12EI}{l^{3}}\Delta$$

• Construct the Stiffness Equation

$$\sum M_B^i = 0 \rightarrow \frac{Pab^2}{l^2} + \frac{8EI}{l} \theta_B + \frac{2EI}{l} \theta_c + \frac{6EI}{l^2} \Delta = 0$$

$$\sum M_C^i = 0 \rightarrow -\frac{Pa^2b}{l^2} + \frac{2EI}{l} \theta_B + \frac{8EI}{l} \theta_c + \frac{6EI}{l^2} \Delta = 0$$

$$\sum V^i = 0 \rightarrow \frac{6EI}{l^2} \theta_B + \frac{6EI}{l^2} \theta_c + \frac{24EI}{l^3} \Delta = 0$$

$$\Delta = -\frac{l}{4} (\theta_B + \theta_C)$$

$$\frac{Pa^{2}b}{l^{2}} + \frac{13EI}{2l}\theta_{B} + \frac{EI}{2l}\theta_{c} = 0$$

$$-\frac{Pab^{2}}{l^{2}} + \frac{EI}{2l}\theta_{B} + \frac{13EI}{2l}\theta_{c} = 0$$

$$\theta_{B} = -\frac{1}{84}\frac{Pab}{EI}\frac{(a+13b)}{l}$$

$$\theta_{C} = \frac{1}{84}\frac{Pab}{EI}\frac{(13a+b)}{l}$$

$$\Delta = \frac{1}{28}\frac{Pab}{EI}(b-a)$$

1.3.5 A Portal Frame with a Bracing (Vertical Load)

i) All fixed

$$M_{BC}^{0} = \frac{Pab^{2}}{l^{2}}$$
, $M_{CB}^{0} = -\frac{Pa^{2}b}{l^{2}}$

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• Construct the Stiffness Equation

$$\begin{split} \sum M_B^i &= 0 \rightarrow \quad \frac{Pab^2}{l^2} + \frac{8EI}{l} \theta_B + \frac{2EI}{l} \theta_c + \frac{6EI}{l^2} \Delta = 0 \\ \sum M_C^i &= 0 \rightarrow \quad -\frac{Pa^2b}{l^2} + \frac{2EI}{l} \theta_B + \frac{8EI}{l} \theta_c + \frac{6EI}{l^2} \Delta = 0 \\ \sum V^i &= 0 \rightarrow \qquad \frac{6EI}{l^2} \theta_B + \frac{6EI}{l^2} \theta_c + \frac{24EI}{l^3} (1+\alpha)\Delta = 0 \\ \Delta &= -\frac{1}{1+\alpha} \frac{l}{4} (\theta_B + \theta_C) \quad , \quad \alpha = \frac{EAl^2}{48\sqrt{2}EI} \end{split}$$

• **Solution for** b = 3a

$$\theta_{B} = -\frac{40 + 52\alpha}{256(7 + 10\alpha)} \frac{Pl^{2}}{EI}, \quad \theta_{C} = -\frac{16 + 28\alpha}{256(7 + 10\alpha)} \frac{Pl^{2}}{EI}, \quad \Delta = \frac{3}{128(7 + 10\alpha)} \frac{Pl^{3}}{EI}$$

For a
$$w \times h$$
 rectangular section and $l = 20h$, $\alpha = 50\sqrt{2}$.
 $\theta_B = -0.0203 \frac{Pl^2}{EI}$, $\theta_C = 0.0109 \frac{Pl^2}{EI}$ $\Delta = 0.3282 \times 10^{-4} \frac{Pl^3}{EI}$

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Response	with Bracing $(\alpha = 50\sqrt{2})$	w/o bracing $(\alpha = 0)$	Ratio(%)
$\theta_B(\times Pl^2/EI)$	-0.0203	-0.0223	91.03
$\theta_C(\times Pl^2/EI)$	0.0109	0.0089	122.47
$\Delta(\times Pl^3 / EI)$	0.3282×10 ⁻⁴	0.0033	0.99
M_{AB} (Pl)	-0.0404	-0.0248	162.90
$M_{BA} (Pl)$	-0.0810	-0.0694	116.71
M_{CD} (Pl)	0.0438	0.0554	79.06
M_{DC} (Pl)	0.0220	0.0376	58.51
M_P (Pl)	0.1158	0.1216	95.23
$A_{BD}(P)$	0.0788	-	-
$P_{\max}(P_{all})^*$	0.0720	0.0685	105.1
P _{max} /vol.	0.0163	0.0228	71.5

Performance

*) $P_{all} = \sigma_{all} wh$, $M_{all} = P_{all} h/6$

Unbalanced shear force in the columns = $\frac{6EI}{I^2}(\theta_B + \theta_C) = 0.0564 P$

The bracing carries 99 % of the unbalanced shear force between the two columns.

1.3.6 A Portal Frame with a Bracing (Horizontal Load)

ii)-iv) the same as the previous case

•

•

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• Construct the Stiffness Equation

$$\sum M_B^i = 0 \rightarrow \frac{8EI}{l} \theta_B + \frac{2EI}{l} \theta_c + \frac{6EI}{l^2} \Delta = 0$$

$$\sum M_C^i = 0 \rightarrow \frac{2EI}{l} \theta_B + \frac{8EI}{l} \theta_c + \frac{6EI}{l^2} \Delta = 0$$

$$\sum V^i = 0 \rightarrow \frac{6EI}{l^2} \theta_B + \frac{6EI}{l^2} \theta_c + \frac{24EI}{l^3} (1+\alpha)\Delta = P$$

$$\theta_B = \theta_C \quad , \quad \Delta = -\frac{5}{3} \theta_B l = -\frac{5}{3} \theta_C l$$

• Solution

$$\theta_B = \theta_C = -\frac{1}{(28+40\alpha)} \frac{Pl^2}{EI}, \ \Delta = \frac{5}{3(28+40\alpha)} \frac{Pl^3}{EI}$$

For $\alpha = 50\sqrt{2}$,

$$\theta_B = \theta_C = -0.3501 \times 10^{-3} \frac{Pl^2}{EI}$$
, $\Delta = 0.5835 \times 10^{-3} \frac{Pl^3}{EI}$

• Performance

Response	with Bracing $(\alpha = 50\sqrt{2})$	w/o bracing $(\alpha = 0)$	Ratio(%)
$\theta_B(\times Pl^2 / EI)$	-0.3501×10^{-3}	-0.3571×10^{-1}	0.98
$\theta_C(\times Pl^2/EI)$	-0.3501×10^{-3}	-0.3571×10^{-1}	0.98
$\Delta(\times Pl^3/EI)$	0.5835×10^{-3}	0.5952×10^{-1}	0.98
M_{AB} (Pl)	0. 0.2801×10^{-2}	0.2857	0.98
M_{BA} (Pl)	1. 0.2101×10^{-2}	0.2143	0.98
M_{CD} (Pl)	0.2101×10^{-2}	0.2143	0.98
M_{DC} (Pl)	0.2801×10^{-2}	0.2857	0.98
$A_{BD}(P)$	1.4004	-	-
$P_{\rm max}(P_{\rm all})^*$	0.7141	0.0292	2448
P _{max} /vol.	0.1617	0.0097	1670

*) Governed by A_{BD} for the structure with bracing, and by M_{DC} for the structure without bracing. $P_{all} = \sigma_{all} wh$, $M_{all} = P_{all} h/6$

The bracing carries about 99% of the external horizontal load.

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1.3.7 A Portal Frame with a Spring

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1.3.8 A Portal Frame Subject to Support Settlement

- **DOF**: θ_B , θ_C , Δ
- Analysis

i) All fixed

$$M_{BC}^{0} = M_{CB}^{0} = \frac{6EI}{l^{2}}\delta$$

ii)-iv) the same as the previous problem

• Construct the Stiffness Equation $\sum M_B^i = 0 \rightarrow \frac{6EI}{l^2} \delta + \frac{8EI}{l} \theta_B + \frac{2EI}{l} \theta_c + \frac{6EI}{l^2} \Delta = 0$ $\sum M_C^i = 0 \rightarrow \frac{6EI}{l^2} \delta + \frac{2EI}{l} \theta_B + \frac{8EI}{l} \theta_c + \frac{6EI}{l^2} \Delta = 0$ $\sum V^i = 0 \rightarrow \frac{6EI}{l^2} \theta_B + \frac{6EI}{l^2} \theta_c + \frac{24EI}{l^3} \Delta = 0$

1.3.9 A Portal Frame with Unsymmetrical Supports

- **DOF**: θ_B , θ_C , Δ
- Analysis

i) All fixed

$$M_{BC}^{0} = \frac{Pl}{8}$$
 , $M_{CB}^{0} = -\frac{Pl}{8}$

ii) $\theta_B \neq 0$

$$M_{BA}^{1} = \frac{3EI}{l} \theta_{B}$$

$$M_{BC}^{1} = \frac{4EI}{l} \theta_{B} , \quad M_{CB}^{1} = \frac{2EI}{l} \theta_{B}$$

$$V_{BA}^{1} = \frac{3EI}{l^{2}} \theta_{B}$$

iii) $\theta_c \neq 0$

$$M_{BC}^{2} = \frac{2EI}{l} \theta_{C} , \quad M_{CB}^{2} = \frac{4EI}{l} \theta_{C}$$
$$M_{CD}^{2} = \frac{4EI}{l} \theta_{C} , \quad M_{DC}^{2} = \frac{2EI}{l} \theta_{C}$$
$$V_{CD}^{2} = \frac{6EI}{l^{2}} \theta_{C}$$

iv) $\Delta \neq 0$

$$M_{BA}^{3} = \frac{3EI}{l^{2}}\Delta ,$$

$$M_{CD}^{3} = M_{DC}^{3} = \frac{6EI}{l^{2}}\Delta ,$$

$$V_{BA}^{3} = \frac{3EI}{l^{3}}\Delta , \quad V_{CD}^{3} = \frac{12EI}{l^{3}}\Delta$$

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• Construct the Stiffness Equation

• Load Location that Causes No Sidesway

- Stiffness equation

$$\sum M_B^i = 0 \rightarrow \frac{Pab^2}{l^2} + \frac{7EI}{l} \theta_B + \frac{2EI}{l} \theta_c = 0 , \quad \sum M_C^i = 0 \rightarrow -\frac{Pa^2b}{l^2} + \frac{2EI}{l} \theta_B + \frac{8EI}{l} \theta_c = 0$$
$$\frac{Pab^2}{l^2} - \frac{12EI}{l} \theta_c = 0 , \quad -\frac{Pa^2b}{l^2} + \frac{4EI}{l} \theta_c = 0$$
$$\frac{Pab^2}{l^2} = 3\frac{Pa^2b}{l^2} \rightarrow b = 3a$$

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1.3.10 A Frame with a Skewed Member

- **DOF**: θ_B , Δ
- Analysis

i) All fixed : $M_{BC}^0 = \frac{Pl}{8} + \frac{Pl}{16} = \frac{3}{16}Pl$, $V_{BC}^0 = -\frac{\sqrt{2}}{2}\frac{11}{16}P$

ii) $\theta_B \neq 0$

iii) $\Delta \neq 0$

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• Construct the stiffness equation

$$\sum M_{\mu}^{i} = 0 \rightarrow (2\sqrt{2} + 3) \frac{EI}{l} \theta_{\mu} + (3 - \frac{3\sqrt{2}}{2}) \frac{EI}{l^{2}} \Delta = -\frac{3}{16} Pl$$

$$\sum V^{i} = 0 \rightarrow (3 - \frac{3}{2}\sqrt{2}) \frac{EI}{l^{2}} \theta_{\mu} + (3\sqrt{2} + \frac{3}{2}) \frac{EI}{l^{2}} \Delta = \frac{\sqrt{2}}{2} \frac{11}{16} P$$

$$= 5.8284 \frac{EI}{l} \theta_{\mu} + 0.8787 \frac{EI}{l^{2}} \Delta = -0.1875 Pl$$

$$= 0.8787 \frac{EI}{l^{2}} \theta_{\mu} + 5.7426 \frac{FI}{l^{2}} \Delta = 0.4861 P$$

$$= \theta_{\mu} = -0.0460 \frac{Pl^{2}}{EI} , \Delta = 0.0917 \frac{Pl^{3}}{EI}$$

$$= \text{Deformed shape}$$

$$= \text{Deformed shape}$$

$$= \text{Omment diagram}$$

$$= \text{Shear force diagram}$$

$$= \text{Shear force diagram}$$

$$= \text{Shear force diagram}$$

Chapter 2

Iterative Solution Method & Moment Distribution Method

2.1 Solution Method for Linear Algebraic Equations

2.1.1 Direct Method – Gauss Elimination

$$\begin{array}{c} a_{11}X_{1} + a_{12}X_{2} + \dots + a_{1i}X_{i} + \dots + a_{1n}X_{n} = b_{1} \\ a_{21}X_{1} + a_{22}X_{2} + \dots + a_{2i}X + \dots + a_{2n}X_{n} = b_{2} \\ \vdots \\ a_{i1}X_{1} + a_{i2}X_{2} + \dots + a_{ii}X_{i} + \dots + a_{in}X_{n} = b_{i} \\ \vdots \\ a_{n1}X_{1} + a_{n2}X_{2} + \dots + a_{ni}X_{i} + \dots + a_{nn}X_{n} = b_{n} \end{array} \right\} \rightarrow \sum_{j=1}^{n} a_{ij}X_{j} = b_{i} \text{ for } i = 1 \dots n$$

or in a matrix form

$$[\mathbf{A}](\mathbf{X}) = (\mathbf{b})$$

By multiplying $\frac{a_{i1}}{a_{11}}$ to the first equation and subtracting the resulting equation from the *i*-th equation for $2 \le i \le n$, the first unknown X_1 is eliminated from the second equation as follows.

$$a_{11}X_{1} + a_{12}X_{2} + \dots + a_{1i}X_{i} + \dots + a_{1n}X_{n} = b_{1}$$

$$a_{22}^{(2)}X_{2} + \dots + a_{2i}^{(2)}X + \dots + a_{2n}^{(2)}X_{n} = b_{2}^{(2)}$$

$$\vdots$$

$$a_{i2}^{(2)}X_{2} + \dots + a_{ii}^{(2)}X_{i} + \dots + a_{in}^{(2)}X_{n} = b_{i}^{(2)}$$

$$\vdots$$

$$a_{n2}^{(2)}X_{2} + \dots + a_{n2}^{(2)}X_{i} + \dots + a_{nn}^{(2)}X_{n} = b_{n}^{(2)}$$

where $a_{ij}^{(2)} = a_{ij} - \frac{a_{i1}a_{1j}}{a_{11}}$. Again, the second unknown X_2 is eliminated from the third equa-

tion by multiplying $\frac{a_{i2}^{(2)}}{a_{22}^{(2)}}$ to the second equation and subtracting the resulting equation from the *i*-th equation for $3 \le i \le n$. The aforementioned procedures are repeated until the last unknown remains in the last equation.

$$\begin{aligned} a_{11}X_1 + a_{12}X_2 &+ \dots + a_{1i}X_i &+ \dots + a_{1n}X_n = b_1 \\ a_{22}^{(2)}X_2 &+ \dots + a_{2i}^{(2)}X &+ \dots + a_{2n}^{(2)}X_n = b_2^{(2)} \\ \vdots & \vdots & \vdots \\ a_{ii}^{(i)}X_i &+ \dots + a_{in}^{(i)}X_n = b_i^{(i)} \\ \vdots & \vdots \\ a_{nn}^{(n)}X_n &= b_n^{(n)} \end{aligned}$$

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where $a_{ij}^{(k)} = a_{ij}^{(k-1)} - \frac{a_{i,k-1}^{(k-1)}a_{k-1,j}^{(k-1)}}{a_{k-1,k-1}^{k-1}}$ $k \le i, j \le n$, and $a_{ij}^1 = a_{ij}$. Once the system matrix is tri-

angularized, the solution of the given system is easily obtained by the back-substitution.

$$a_{nn}^{(n)} X_{n} = b_{n}^{(n)} \rightarrow \qquad X_{n} = \frac{b_{n}^{(n)}}{a_{nn}^{(n)}}$$

$$a_{n-1,n-1}^{(n-1)} X_{n-1} + a_{n-1,n}^{(n-1)} X_{n} = b_{n-1}^{(n-1)} \rightarrow \qquad X_{n-1} = \frac{b_{n-1}^{(n-1)} - a_{n-1,n}^{(n-1)} X_{n}}{a_{n-1,n-1}^{n-2}}$$

$$a_{ii}^{(i)} X_{i} + \dots + a_{i,n}^{(i)} X_{n} = b_{i}^{(i)} \rightarrow \qquad X_{i} = \frac{b_{i}^{(i)} - \sum_{k=n}^{i+1} a_{ik}^{(i)} X_{k}}{a_{ii}^{i-1}} \quad \text{for } 1 \le i \le n-1$$

2.1.2 Iterative Method – Gauss-Jordan Method

A system of linear algebraic equations may be solved by iterative method. For this purpose, the given system is rearranged as follows.

$$X_{1} = \frac{b_{1} - (a_{12}X_{2} + \dots + a_{1n}X_{n})}{a_{11}}$$

$$X_{2} = \frac{b_{2} - (a_{21}X_{1} + a_{23}X_{3} + \dots + a_{2n}X_{n})}{a_{22}}$$

$$X_{i} = \frac{b_{i} - (a_{i1}X_{1} + \dots + a_{i,i-1}X_{i-1} + a_{i,i+1}X_{i-1} + \dots + a_{in}X_{n})}{a_{ii}}$$

$$X_{n} = \frac{b_{n} - (a_{n1}X_{1} + \dots + a_{n,n-1}X_{n-1})}{a_{nn}}$$

Suppose we substitute an approximate solution $(\mathbf{X})_{k-1}$ into the right-hand side of the above equation, a new approximate solution $(\mathbf{X})_k$, which is not the same as $(\mathbf{X})_{k-1}$, is obtained. This procedure is repeated until the solution converges.

$$(X_i)_k = \frac{1}{a_{ii}} (b_i - \sum_{\substack{j=1\\i\neq j}}^n a_{ij} (X_j)_{k-1})$$

where the subscript k denotes the iterational count.

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2.1.3 Iterative Method – Gauss-Siedal Method

When we calculate a new X_i value in the *k*-th iteration of Gauss-Jordan iteration, the values of X_1, \dots, X_{i-1} are already updated, and we can utilize the updated values to accelerate convergence rate, which leads to the Gauss-Siedal Method.

$$(X_{1})_{k} = \frac{b_{1} - (a_{12}(X_{2})_{k-1} + \dots + a_{1n}(X_{n})_{k-1})}{a_{11}}$$

$$(X_{2})_{k} = \frac{b_{2} - (a_{21}(X_{1})_{k} + a_{23}X_{3})_{k-1} + \dots + (a_{2n}X_{n})_{k-1})}{a_{22}}$$

$$(X_{i})_{k} = \frac{b_{i} - (a_{i1}(X_{1})_{k} + \dots + a_{i,i-1}(X_{i-1})_{k} + a_{i,i+1}(X_{i+1})_{k-1} + \dots + a_{in}(X_{n})_{k-1})}{a_{ii}}$$

$$(X_{n})_{k} = \frac{b_{n} - (a_{n1}(X_{1})_{k} + \dots + a_{n,n-1}(X_{n-1})_{k})}{a_{nn}}$$

$$(X_i)_k = \frac{1}{a_{ii}} (b_i - \sum_{\substack{j=1\\i>1}}^{i-1} a_{ij} (X_j)_k - \sum_{\substack{j=i+1\\i< n}}^n a_{ij} (X_j)_{k-1})$$

2.1.4 Example

$$A \qquad B \qquad C \qquad 4klf \qquad D$$

$$A \qquad EI \qquad 90 \qquad 1.5EI \qquad 90 \qquad EI \qquad 90 \qquad 0$$

• Stiffness Equation

$$-168.7 + 133.3 + (\frac{3EI}{l} + \frac{6EI}{l})\theta_{B} + \frac{3EI}{l}\theta_{C} = 0$$

$$-133.3 + 450.0 + \frac{3EI}{l}\theta_{B} + (\frac{6EI}{l} + \frac{3EI}{l})\theta_{C} = 0$$

For the simplicity of derivation, $\frac{EI}{l}\theta_B \rightarrow \theta_B$, $\frac{EI}{l}\theta_C \rightarrow \theta_C$. The stiffness equation becomes $-35.4 + 9\theta_B + 3\theta_C = 0$ $316.7 + 3\theta_B + 9\theta_C = 0$

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• Gauss-Jordan Iteration

$$(\theta_B)_k = \frac{1}{9}(35.4 - 3(\theta_C)_{k-1})$$

$$(\theta_C)_k = \frac{1}{9}(-316.7 - 3(\theta_B)_{k-1})$$

• Gauss-Siedal Iteration

$$(\theta_B)_k = \frac{1}{9}(35.4 - 3(\theta_C)_{k-1})$$
$$(\theta_C)_k = \frac{1}{9}(-316.7 - 3(\theta_B)_k)$$

Gauss-Seidal	Gauss-Jordan				
GAUSS-SIEDAL ITERATION	GAUSS-Jordan ITERATION				
***** Iteration 1*****	***** Iteration 1*****				
X(1) = 0.393334E+01	X(1) = 0.393334E+01				
X(2) = -0.3650000E+02	X(2) = -0.3518889E+02				
ERROR = 0.100000E+01	ERROR = 0.100000E+01				
***** Iteration 2*****	***** Iteration 2*****				
X(1) = 0.1610000E+02	X(1) = 0.1566296E+02				
X(2) = -0.4055556E+02	X(2) = -0.3650000E+02				
ERROR = 0.2939146E+00	ERROR = 0.2971564E+00				
***** Iteration 3*****	***** Iteration 3*****				
X(1) = 0.1745185E+02	X(1) = 0.1610000E+02				
X(2) = -0.4100617E+02	X(2) = -0.4040988E+02				
ERROR = 0.3197497E-01	ERROR = 0.9044394E-01				
***** Iteration 4*****	***** Iteration 4*****				
X(1) = 0.1760206E+02	X(1) = 0.1740329E+02				
X(2) = -0.4105624E+02	X(2) = -0.4055556E+02				
ERROR = 0.3544420E - 02	ERROR = 0.2971564E-01				
***** Iteration 5*****	***** Iteration 5 ** ***				
X(1) = 0.1761875E+02	X(1) = 0.1745185E+02				
X(2) = -0.4106181E+02	X(2) = -0.4098999E+02				
ERROR = 0.3937214E-03	ERROR = 0.9812154E-02				
***** MOMENT *****	***** MOMENT *****				
MBA =-115.84375	MBA =-116.34444				
MBC = 115.82707	MBC = 115.04115				
MCB =-326.81460	MCB =-326.88437				
MCD = 326.81458	MCD = 327.03004				

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2.2 Moment Distribution Method



• At the Joint B

- Moment distribution

$$M_{B} = (\frac{3EI_{AB}}{L_{AB}} + \frac{4EI_{BC}}{L_{BC}})\theta_{B} = M_{BA}^{1} + M_{BC}^{1} \rightarrow \theta_{B} = \frac{M_{B}}{\frac{3EI_{AB}}{L_{AB}} + \frac{4EI_{BC}}{L_{BC}}}$$

$$M_{BA}^{1} = \frac{3EI_{AB}}{L_{AB}} \theta_{B} = \frac{\frac{3EI_{AB}}{L_{AB}}}{\frac{3EI_{AB}}{L_{AB}}} M_{B} = D_{BA}M_{B}$$

$$M_{BC}^{1} = \frac{4EI_{BC}}{L_{BC}} \theta_{B} = \frac{\frac{4EI_{BC}}{L_{BC}}}{\frac{3EI_{AB}}{L_{AB}} + \frac{4EI_{BC}}{L_{BC}}} M_{B} = D_{BC}M_{B}$$
The equation is integrable of $M_{AB}^{1} = \frac{2EI_{BC}}{L_{BC}} = \frac{1}{2}D_{AB}M_{B}$

- Moment carry over to joint C: $M_{CB}^1 = \frac{2EI_{BC}}{L_{BC}} \theta_B = \frac{1}{2} D_{BC} M_B$

• At the Joint C

- Moment distribution

$$M_{C} = \left(\frac{4EI_{BC}}{L_{BC}} + \frac{3EI_{CD}}{L_{CD}}\right)\theta_{C} = M_{CB}^{2} + M_{CD}^{2} \rightarrow \theta_{C} = \frac{M_{C}}{\frac{4EI_{BC}}{L_{BC}} + \frac{3EI_{CD}}{L_{CD}}}$$
$$M_{CB}^{2} = \frac{4EI_{BC}}{L_{BC}}\theta_{C} = \frac{\frac{4EI_{BC}}{L_{BC}}}{\frac{4EI_{BC}}{L_{BC}} + \frac{3EI_{CD}}{L_{CD}}}M_{C} = D_{CB}M_{C}$$
$$M_{CD}^{2} = \frac{3EI_{CD}}{L_{CD}}\theta_{C} = \frac{\frac{3EI_{CD}}{L_{CD}}}{\frac{4EI_{BC}}{L_{BC}} + \frac{3EI_{CD}}{L_{CD}}}M_{C} = D_{CD}M_{C}$$

- Moment carry over to joint B: $M_{BC}^2 = \frac{2EI_{BC}}{L_{BC}} \theta_C = \frac{1}{2} D_{CB} M_C$

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• Stiffness Equation in terms of Moment at Joints

$$-35.4 + M_{B} + \frac{1}{2}D_{CB}M_{C} = 0$$

$$M_{B} = 35.4 - \frac{1}{2}D_{CB}M_{C}$$

$$M_{B} = 35.4 - \frac{1}{2}D_{CB}M_{C}$$

$$M_{C} = -316.7 - \frac{1}{2}D_{BC}M_{B}$$

- Gauss-Seidal Approach

$$(M_B)_k = 35.4 - \frac{1}{2} D_{CB} (M_C)_{k-1}$$
$$(M_C)_k = -316.7 - \frac{1}{2} D_{BC} (M_B)_k$$

- Gauss-Jordan Approach

$$(M_B)_k = 35.4 - \frac{1}{2} D_{CB} (M_C)_{k-1}$$
$$(M_C)_k = -316.7 - \frac{1}{2} D_{BC} (M_B)_{k-1}$$

• For the given structure

$$D_{BA} = D_{CD} = \frac{1}{3}$$
, $D_{BC} = D_{CB} = \frac{2}{3}$

• Incremental form for the Gauss-Seidal Method

- For k = 1

 $(M_B)_0 = (M_C)_0 = 0$ because we assume all degrees of freedom are fixed for step 0.

$$(M_B)_1 = 35.4 - \frac{1}{2} D_{CB} (M_C)_0 = 35.4 \rightarrow (\Delta M_B)_1 = 35.4$$
$$(M_C)_1 = -316.7 - \frac{1}{2} D_{BC} (M_B)_1 = -316.7 - \frac{1}{2} \frac{2}{3} 35.4 \rightarrow (\Delta M_C)_1 = -328.5$$

- For k > 1

$$(M_{B})_{k} = 35.4 - \frac{1}{2} D_{CB} (M_{c})_{k-1} = 35.4 - \frac{1}{2} D_{CB} (M_{c})_{k-2} - \frac{1}{2} D_{CB} (\Delta M_{c})_{k-1}$$
$$= (M_{B})_{k-1} - \frac{1}{2} D_{CB} (\Delta M_{c})_{k-1} \rightarrow (\Delta M_{B})_{k} = -\frac{1}{2} D_{CB} (\Delta M_{c})_{k-1}$$
$$(M_{C})_{k} = -316.7 - \frac{1}{2} D_{BC} (M_{B})_{k} = -316.7 - \frac{1}{2} D_{BC} (M_{B})_{k-1} - \frac{1}{2} D_{BC} (\Delta M_{B})_{k}$$
$$= (M_{C})_{k-1} - \frac{1}{2} D_{BC} (\Delta M_{B})_{k} \rightarrow (\Delta M_{C})_{k} = -\frac{1}{2} D_{BC} (\Delta M_{B})_{k}$$

- Iteration 1

$$\begin{split} (M_B)_0 &= 0, \ (M_C)_0 = 0 \\ (M_B)_1 &= 35.4 - \frac{1}{2} D_{CB} (M_C)_0 = 35.4 \rightarrow (\Delta M_B)_1 = 35.4 \\ (M_{BA})_1 &= M_{BA}^f + D_{BA} (\Delta M_B)_1 = -168.7 + 0.33 \times 35.4 = -168.7 + 11.8 \rightarrow (\Delta M_{BA})_1 = 11.8 \\ (M_{BC})_1 &= M_{BC}^f + D_{BC} (\Delta M_B)_1 = 133.3 + 0.67 \times 35.4 = 133.3 + 23.6 \rightarrow (\Delta M_{BC})_1 = 23.6 \\ (M_C)_1 &= -316.7 - \frac{1}{2} D_{BC} (\Delta M_B)_1 = -316.7 - \frac{1}{2} \frac{2}{3} 35.4 = -328.5 \rightarrow (\Delta M_C)_1 = -328.5 \\ (M_{CB})_1 &= M_{CB}^f + \frac{1}{2} D_{BC} (\Delta M_B)_1 + D_{CB} (\Delta M_C)_1 = -133.3 + 11.8 - 219.0 \rightarrow (\Delta M_{CB})_1 = 11.8 - 219.0 \\ (M_{CD})_1 &= M_{CD}^f + D_{CD} (\Delta M_C)_1 = 450 - 0.33 \times 328.5 = 450 - 109.5 \rightarrow (\Delta M_{CD})_1 = -109.5 \\ \end{split}$$

- Iteration 2

$$\begin{split} (\Delta M_B)_2 &= -\frac{1}{2} D_{CB} (\Delta M_c)_1 = 109.5 \rightarrow \begin{cases} (\Delta M_{BA})_2 = D_{BA} (\Delta M_B)_2 = 36.5 \\ (\Delta M_{BC})_2 &= \frac{1}{2} D_{CB} (\Delta M_c)_1 + D_{BC} (\Delta M_B)_2 \\ &= -109.5 + 73.0 \end{cases} \\ (\Delta M_C)_2 &= -\frac{1}{2} D_{BC} (\Delta M_B)_2 = -36.5 \rightarrow \begin{cases} (\Delta M_{CB})_2 = \frac{1}{2} D_{BC} (\Delta M_B)_2 + D_{CB} (\Delta M_C)_2 \\ &= 36.5 - 24.3 \\ (\Delta M_{CD})_2 = D_{CD} (\Delta M_C)_2 = -12.2 \end{cases} \end{split}$$

- Iteration 3

$$(\Delta M_B)_3 = -\frac{1}{2} D_{CB} (\Delta M_c)_2 = 12.2 \rightarrow \begin{cases} (\Delta M_{BA})_3 = D_{BA} (\Delta M_B)_3 = 4.1 \\ (\Delta M_{BC})_3 = \frac{1}{2} D_{CB} (\Delta M_c)_2 + D_{BC} (\Delta M_B)_3 \\ = -12.2 + 8.2 \end{cases}$$
$$(\Delta M_C)_3 = -\frac{1}{2} D_{BC} (\Delta M_B)_3 = -4.1 \rightarrow \begin{cases} (\Delta M_{CB})_3 = \frac{1}{2} D_{BC} (\Delta M_B)_3 + D_{CB} (\Delta M_C)_3 = 4.1 - 2.7 \\ (\Delta M_{CD})_3 = D_{CD} (\Delta M_C)_3 = -1.4 \end{cases}$$

- Final Moments

$$\begin{split} M_{BA} &= M_{BA}^{f} + \sum_{k} (\Delta M_{BA})_{k} = -168.7 + 11.8 + 36.5 + 4.1 = -116.3 \\ M_{BC} &= M_{BC}^{f} + \sum_{k} (\Delta M_{BC})_{k} = 133.3 + 23.6 + (-109.5 + 73.0) + (-12.2 + 8.2) = 116.4 \\ M_{CB} &= M_{CB}^{f} + \sum_{k} (\Delta M_{CB})_{k} = -133.3 + (11.8 - 219.0) + (36.5 - 24.3) + (4.1 - 2.7) = -326.9 \\ M_{CD} &= M_{CD}^{f} + \sum_{k} (\Delta M_{CD})_{k} = 450.0 - 109.5 - 12.2 - 1.4 = 326.9 \end{split}$$

------ Structural Analysis Lab.

	0.33	0.66	(0.67	0.33			
A.		-168.7	133.3		-133.3	450.0		
		11.8	23.6		→ 11.8			
			-109.5	•	-219.0	-109.5		
		36.5	73.0		→ 36.5		-	
			-12.2	•	-24.3	-12.2	_	
		4.1	8.2		→ 4.1			
			-1.4		-2.7	-1.4		
		0.5	0.9		-326.9	326.9	-	
		-115.8	115.9	_				

• Incremental form for the Gauss-Jordan Method

- For k = 1

 $(M_B)_0 = (M_C)_0 = 0$ because we assume all degrees of freedom are fixed for step 0.

$$(M_B)_1 = 35.4 - \frac{1}{2} D_{CB} (M_c)_0 = 35.4 \rightarrow (\Delta M_B)_1 = 35.4$$

 $(M_C)_1 = -316.7 - \frac{1}{2} D_{BC} (M_B)_0 = -316.7 \rightarrow (\Delta M_C)_1 = -316.7$

- For k > 1

$$(M_{B})_{k} = 35.4 - \frac{1}{2} D_{CB} (M_{c})_{k-1} = 35.4 - \frac{1}{2} D_{CB} (M_{c})_{k-2} - \frac{1}{2} D_{CB} (\Delta M_{c})_{k-1}$$
$$= (M_{B})_{k-1} - \frac{1}{2} D_{CB} (\Delta M_{c})_{k-1} \rightarrow (\Delta M_{B})_{k} = -\frac{1}{2} D_{CB} (\Delta M_{c})_{k-1}$$
$$(M_{C})_{k} = -316.7 - \frac{1}{2} D_{BC} (M_{B})_{k-1} = -316.7 - \frac{1}{2} D_{BC} (M_{B})_{k-2} - \frac{1}{2} D_{BC} (\Delta M_{B})_{k-1}$$
$$= (M_{C})_{k-1} - \frac{1}{2} D_{BC} (\Delta M_{B})_{k-1} \rightarrow (\Delta M_{C})_{k} = -\frac{1}{2} D_{BC} (\Delta M_{B})_{k-1}$$

- Iteration 1

$$(M_B)_0 = 0, \ (M_C)_0 = 0$$

$$(M_B)_1 = 35.4 - \frac{1}{2} D_{CB} (M_C)_0 = 35.4 \rightarrow (\Delta M_B)_1 = 35.4$$

$$(M_{BA})_1 = M_{BA}^f + D_{BA} (M_B)_1 = -168.7 + 0.33 \times 35.4 = -168.7 + 11.8 \rightarrow (\Delta M_{BA})_1 = 11.8$$

$$(M_{BC})_1 = M_{BC}^f + D_{BC} (M_B)_1 = 133.3 + 0.67 \times 35.4 = 133.3 + 23.6 \rightarrow (\Delta M_{BC})_1 = 23.6$$

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$$(M_{c})_{1} = -316.7 - \frac{1}{2} D_{BC} (M_{B})_{0} = -316.7 = -316.7 \rightarrow (\Delta M_{c})_{1} = -316.7$$
$$(M_{CB})_{1} = M_{CB}^{f} + D_{CB} (M_{C})_{1} = -133.3 - 0.67 \times 316.7 = -133.3 - 212.2 \rightarrow (\Delta M_{CB})_{1} = -212.2$$
$$(M_{CD})_{1} = M_{CD}^{f} + D_{CD} (M_{C})_{1} = 450 - 0.33 \times 316.7 = 450 - 104.5 \rightarrow (\Delta M_{CD})_{1} = -104.5$$

- Iteration 2

$$(\Delta M_B)_2 = -\frac{1}{2} D_{CB} (\Delta M_c)_1 = 106.1 \rightarrow \begin{cases} (\Delta M_{BA})_2 = D_{BA} (\Delta M_B)_2 = 35.0 \\ (\Delta M_{BC})_2 = \frac{1}{2} D_{CB} (\Delta M_c)_1 + D_{BC} (\Delta M_B)_2 \\ = -106.1 + 71.1 \end{cases}$$
$$(\Delta M_C)_2 = -\frac{1}{2} D_{BC} (\Delta M_B)_1 = -11.8 \rightarrow \begin{cases} (\Delta M_{CB})_2 = \frac{1}{2} D_{BC} (\Delta M_B)_1 + D_{CB} (\Delta M_C)_2 \\ = 11.8 - 7.9 \\ (\Delta M_{CD})_2 = D_{CD} (\Delta M_C)_2 = -3.9 \end{cases}$$

- Iteration 3

$$(\Delta M_B)_3 = -\frac{1}{2} D_{CB} (\Delta M_C)_2 = 4.0 \rightarrow \begin{cases} (\Delta M_{BA})_3 = D_{BA} (\Delta M_B)_3 = 1.3 \\ (\Delta M_{BC})_3 = \frac{1}{2} D_{CB} (\Delta M_C)_2 + D_{BC} (\Delta M_B)_3 \\ = -4.0 + 2.7 \end{cases}$$
$$(\Delta M_C)_3 = -\frac{1}{2} D_{BC} (\Delta M_B)_2 = -35.6 \rightarrow \begin{cases} (\Delta M_{CB})_3 = \frac{1}{2} D_{BC} (\Delta M_B)_3 + D_{CB} (\Delta M_C)_3 \\ = 35.6 - 23.9 \\ (\Delta M_{CD})_3 = D_{CD} (\Delta M_C)_3 = -11.7 \end{cases}$$

	0.33	0.66	0.0	67 0.33		
Â		-168.7	133.3	-133.3 🖉	§ 450.0	Â
		11.8	23.6	-212.2	-104.5	
			-106.1	11.8		
		35.0	71.1 -	-7.9	-3.9	
			-4.0	35.6		
		1.3	2.7	-23.9	-11.7	
			-12.0	1.4		
		4.0	8.0 -	-0.9	-0.5	
			-0.5	4.0		
		0.2	0.3	-2.7	-1.3	
			-1.4	0.2		
		0.5	0.9	0.1	0.1	
		-115.9	115.9	-328.0	328.2	
		<u>0.5</u> -115.9	-1.4 0.9 115.9	0.2 0.1 -328.0	0.1 328.2	_

_____ Structural Analysis Lab.

2.3 Example - MDM for a 4-span Continuous Beam





$$D_{CD} = 4\frac{1.5EI}{L} \left/ \left(4\frac{1.5EI}{L} + 4\frac{1.5EI}{L}\right) = 0.5, \quad D_{DC} = 4\frac{1.5EI}{L} \left/ \left(4\frac{1.5EI}{L} + 3\frac{EI}{L}\right) = 0.67, \quad D_{DE} = 3\frac{EI}{L} \left/ \left(4\frac{1.5EI}{L} + 3\frac{EI}{L}\right) = 0.33$$

• Gauss-Siedal Approach



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• Gauss-Jordan Approach

8	0.4	0.6	0.5	0.5		0.67	0.33	
0.0	0.0	125	-125 -	0.0		0.0	93.8	
	-50	-75	62.5	62.5	_>	-62.8	-31.0	
-25.0		31.3	-37.5	-31.4		31.3		
	-12.5	-18.8	34.5	34.5		-21.0	-10.3	
-6.3		17.3	-9.4	-10.5		17.3		
	-6.9	-10.4	10.0	10.0		-11.6	-5.7	
-3.5		5.0	-5.2	-5.8		5.0		
	-2.0	-3.0	5.5	5.5	_	-3.4	-1.6	
-1.0		2.8	-1.5	-1.7		2.8		
	-1.1	-1.7	1.6	1.6		-1.9	-0.9	
-0.6		0.8	-0.9	-1.0		0.8		
	-0.3	-0.5	1.0	1.0		-0.5	-0.3	
-36.4	-72.8	72.8	-64.4	64.7		-44.0	44.0	

2.4 Direct Solution Scheme by Partitioning



• Slope deflection (Stiffness) Equation

$$\begin{bmatrix} \frac{8EI}{L} & \frac{2EI}{L} & \frac{6EI}{L} \\ \frac{2EI}{L} & \frac{8EI}{L} & \frac{6EI}{L} \\ \frac{6EI}{L} & \frac{6EI}{L} & \frac{24EI}{L} \end{bmatrix} \begin{pmatrix} \theta_B \\ \theta_C \\ \Delta \end{pmatrix} + \begin{pmatrix} 88.8 \\ -44.4 \\ 0.0 \end{pmatrix} = 0 \rightarrow \begin{bmatrix} [\mathbf{K}_{\theta\theta}] & (\mathbf{K}_{\theta\Delta}) \\ (\mathbf{K}_{\Delta\theta}) & \mathbf{K}_{\Delta\Delta} \end{bmatrix} \begin{pmatrix} (\Theta) \\ \Delta \end{pmatrix} + \begin{pmatrix} (\mathbf{P}) \\ 0 \end{pmatrix} = 0$$

$$[\mathbf{K}_{\theta\theta}](\Theta) = -(\mathbf{P}) - (\mathbf{K}_{\theta\Delta})\Delta \rightarrow (\Theta) = -[\mathbf{K}_{\theta\theta}]^{-1}(\mathbf{P}) - [\mathbf{K}_{\theta\theta}]^{-1}(\mathbf{K}_{\theta\Delta})\Delta = (\Theta)^{P} + (\Theta)^{\Delta}$$
$$(\mathbf{K}_{\Delta\theta})(\Theta)^{P} + (\mathbf{K}_{\Delta\theta})(\Theta)^{\Delta} + \mathbf{K}_{\Delta\Delta}\Delta = 0$$

• Direct Solution Procedure by Partitioning

- Assume $\Delta = 0$ and calculate $(\Theta)^{P}$.
- Assume an arbitrary $\overline{\Delta} = \frac{\Delta}{\alpha}$ and calculate $(\overline{\Theta})^{\Delta}$.

- By linearity,
$$(\overline{\Theta})^{\Delta} = \frac{(\Theta)^{\Delta}}{\alpha}$$

- Calculate α by the second equation by $\overline{\Delta}$ and $(\overline{\Theta})^{\Delta}$.

$$\alpha = -\frac{(\mathbf{K}_{\Delta\theta})(\Theta)^{P}}{\mathbf{K}_{\Delta\Delta}\overline{\Delta} + (\mathbf{K}_{\Delta\theta})(\overline{\Theta})^{\Delta}}$$

- Obtain (Θ) by (Θ) = (Θ)^{*P*} + α ($\overline{\Theta}$)^{Δ}.

—— Structural Analysis Lab.

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2.5 Moment Distribution Method for Frames

- Solution Procedure
 - Assume there is no sideway and do the MDM.
 - Perform the MDM again for an assumed sidesway.
 - Adjust the Moment obtained by the second MDM to satisfy the second equation.
 - Add the adjusted moment to the moment by the first MDM.

• First MDM – with no Sidesway



 $M_{AB} = -53.3/2 = -26.7, M_{DC} = -35.5/2 = 17.8, V_{AB} = 2.67, V_{DC} = -1.78, V^P = 0.89$

• Second MDM – with an Arbitrary Sidesway



 $M_{AB} = 10 - 3.9/2.0 = 8.1, M_{DC} = 10 - 4.1/2.0 = 7.9, V_{AB} = -0.47, V_{DC} = -0.46, V^{\overline{\Delta}} = -0.93$

Structural Analysis Lab.



• Shear Equilibrium Condition (the 2nd equation)

 $\alpha = -0.89/(-0.93) = 0.97$

Total Moment = 1^{st} Moment + $\alpha 2^{nd}$ Moment

Chapter 3

Buckling of Structures



3.0 Stability of Structures



• Stable state

$$K\Theta L \times L > P\Theta L \rightarrow KL > P$$

The structure would return its original equilibrium position for a small perturbation in θ .



• Unstable state

Critical state

•

$$K \Theta L \times L < Q \Theta L \longrightarrow K L < P$$

The structure would not return its original equilibrium position for a small perturbation in θ .



— Structural Analysis Lab.

3.1 Governing Equation for a Beam with Axial Force



• Equilibrium for vertical force

$$(V+dV) - V + qdx = 0 \rightarrow \frac{dV}{dx} = -q$$

• Equilibrium for moment

$$(M + dM) - M - Vdx + qdx\frac{dx}{2} - P\frac{dw}{dx}dx = 0 \rightarrow \frac{dM}{dx} - P\frac{dw}{dx} = V$$

• Elimination of shear force

$$\frac{d^2M}{dx^2} - P\frac{d^2w}{dx^2} = -q$$

• Strain-displacement relation

$$\varepsilon = -\frac{d^2 w}{dx^2} y + \frac{du}{dx}$$

• Stress-strain relation (Hooke law)

$$\sigma = E\varepsilon = -E\frac{d^2w}{dx^2}y + E\frac{du}{dx}$$

• Definition of Moment

$$M = \int_{A} \sigma y dA = \int_{A} E \varepsilon y dA = -\int_{A} \left(E \frac{d^2 w}{dx^2} y^2 - E \frac{du}{dx} y \right) dA = -EI \frac{d^2 w}{dx^2}$$

• Beam Equation with Axial Force

$$EI\frac{d^4w}{dx^4} + P\frac{d^2w}{dx^2} = q$$

— Structural Analysis Lab.

3.2 Homogeneous Solutions

• **Characteristic Equation for** P > 0

$$w = e^{\lambda x}$$

$$e^{\lambda x} (\lambda^4 + \beta^2 \lambda^2) = 0 \rightarrow \lambda = \pm \beta i, 0$$
 where $\beta^2 = \frac{P}{EI}$

• Homogeneous solution

$$w = Ae^{\beta ix} + Be^{-\beta ix} + Cx + D$$

• Exponential Function with Complex Variable

$$e^{ix} = 1 + ix + \frac{i^2}{2!}x^2 + \frac{i^3}{3!}x^3 + \frac{i^4}{4!}x^4 + \frac{i^5}{5!}x^5 + \frac{i^6}{6!}x^6 + \cdots$$

$$e^{-ix} = 1 - ix + \frac{(-i)^2}{2!}x^2 + \frac{(-i)^3}{3!}x^3 + \frac{(-i)^4}{4!}x^4 + \frac{(-i)^5}{5!}x^5 + \frac{(-i)^6}{6!}x^6 + \cdots$$

$$e^{ix} + e^{-ix} = 2(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots) = 2\cos x$$

$$e^{ix} - e^{-ix} = 2(ix - \frac{i}{3!}x^3 + \frac{i}{5!}x^5 - \frac{i}{7!}x^7 + \cdots) = 2i\sin x$$

 $e^{ix} = \cos x + i \sin x$, $e^{-ix} = \cos x - i \sin x$

• Homogeneous solution

$$w = A(\cos\beta x + i\sin\beta x) + B(\cos\beta x - i\sin\beta x) + Cx + D$$
$$= (A + B)\cos\beta x + i(A - B)\sin\beta x + Cx + D$$
$$= A\cos\beta x + B\sin\beta x + Cx + D$$

• Characteristic Equation for P < 0

$$w = e^{\lambda x}$$

$$e^{\lambda x} (\lambda^4 - \beta^2 \lambda^2) = 0 \rightarrow \lambda = \pm \beta, 0$$
 where $\beta^2 = \left| \frac{P}{EI} \right|$

• Homogeneous solution for P < 0

$$w = Ae^{\beta x} + Be^{-\beta x} + Cx + D$$

= $(A+B)\frac{e^{\beta x} + e^{-\beta x}}{2} + (A-B)\frac{e^{\beta x} - e^{-\beta x}}{2} + Cx + D$
= $A\cosh\beta x + B\sinh\beta x + Cx + D$

• Simple Beam



- Boundary Condition

$$w(0) = A + D = 0$$
, $w''(0) = -A\beta^2 = 0 \rightarrow A = 0$
 $w(L) = B\sin\beta L + CL = 0$, $w''(L) = -B\beta^2\sin\beta L = 0 \rightarrow B = C = 0$

- Characteristic Equation

$$A = B = C = D = 0 \rightarrow w = 0 \quad \text{(trivial solution) or}$$
$$\beta L = n\pi \rightarrow P = \frac{n^2 \pi^2 EI}{L^2} , \quad n = 1, 2, 3 \cdots$$
$$w = B \sin \beta x = B \sin \frac{n\pi}{L} x$$

• Fixed-Fixed Beam



- Boundary Condition

$$w(0) = A + D = 0$$

$$w'(0) = \beta B + C = 0$$

$$w(L) = A\cos\beta L + B\sin\beta L + CL + D = 0$$

$$w'(L) = -A\beta\sin\beta L + B\beta\cos\beta L + C = 0$$

- Characteristic Equation

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & \beta & 1 & 0 \\ \cos\beta L & \sin\beta L & L & 1 \\ -\beta\sin\beta L & \beta\cos\beta L & 1 & 0 \end{bmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow Det \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & \beta & 1 & 0 \\ -\beta\sin\beta L & \beta\cos\beta L & 1 & 0 \end{bmatrix} = 0$$
$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & \beta & 1 & 0 \\ \cos\beta L & \sin\beta L & L & 1 \\ -\beta\sin\beta L & \beta\cos\beta L & 1 & 0 \end{bmatrix} = \begin{vmatrix} \beta & 1 & 0 \\ \sin\beta L & L & 1 \\ \beta\cos\beta L & 1 & 0 \end{vmatrix} = \begin{vmatrix} \beta & 1 & 0 \\ \sin\beta L & L & 1 \\ \beta\cos\beta L & 1 & 0 \end{vmatrix} - \begin{vmatrix} 0 & \beta & 1 \\ \cos\beta L & \sin\beta L & L \\ -\beta\sin\beta L & \beta\cos\beta L & 1 \end{vmatrix}$$

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$$-\beta - (-\beta \cos\beta L) - (-\beta (\cos\beta L + \beta L \sin\beta L) + \beta) = \beta (2\cos\beta L - 2 + \beta L \sin\beta L) = 0$$

$$2\cos\beta L - 2 + \beta L \sin\beta L = 2(\cos\beta L - 1) + \beta L \sin\beta L =$$

$$-4\sin^2 \frac{\beta L}{2} + 2\beta L \sin \frac{\beta L}{2} \cos \frac{\beta L}{2} =$$

$$\sin \frac{\beta L}{2} (\frac{\beta L}{2} \cos \frac{\beta L}{2} - \sin \frac{\beta L}{2}) = 0 \rightarrow \sin \frac{\beta L}{2} = 0 \text{ or } \frac{\beta L}{2} \cos \frac{\beta L}{2} - \sin \frac{\beta L}{2} = 0$$

- Eigenvalues

Symmetric modes

$$\sin \frac{\beta L}{2} = 0 \to \frac{\beta L}{2} = n\pi \to P = \frac{4n^2 \pi^2 EI}{L^2}, \quad n = 1, 2, 3 \cdots$$
$$w(0) = A + D = 0, \quad w'(0) = w'(L) = \beta B + C = 0, \quad w(L) = A + CL + D = 0$$
$$A + D = 0 \to A = -D \to w = A(\cos \frac{2n\pi}{L}x - 1) \quad \text{for} \quad A \neq 0$$

Anti-symmetric modes

$$\frac{\beta L}{2}\cos\frac{\beta L}{2} - \sin\frac{\beta L}{2} = 0 \rightarrow \frac{\beta L}{2} = \tan\frac{\beta L}{2} \rightarrow P = \frac{8.18\pi^2 EI}{L^2}$$
$$D = -A , \quad C = -\beta B$$
$$-A\beta\sin\beta L + B\beta\cos\beta L - \beta B = -2A\beta\sin\frac{\beta L}{2}\cos\frac{\beta L}{2} - 2\beta B\sin^2\frac{\beta L}{2}$$

$$= -2\beta \sin \frac{\beta L}{2} \cos \frac{\beta L}{2} (A + B \tan \frac{\beta L}{2})$$
$$= -2\beta \sin \frac{\beta L}{2} \cos \frac{\beta L}{2} (A + B \frac{\beta L}{2}) = 0 \rightarrow B = -\frac{2}{\beta L} A$$
$$C = -\beta B = \frac{2}{L} A$$

$$A\cos\beta L + B\sin\beta L + CL + D = = A(\cos\beta L - 1 - \frac{2}{\beta L}\sin\beta L + 2)$$

= $A(-2\sin^2\frac{\beta L}{2} - \frac{4}{\beta L}\sin\frac{\beta L}{2}\cos\frac{\beta L}{2} + 2)$
= $A(-2\sin^2\frac{\beta L}{2} - 2\cos^2\frac{\beta L}{2} + 2) = 0$
 $w = A\cos\beta x + B\sin\beta x + Cx + D = A(\cos\beta x - \frac{2}{\beta L}\sin\beta x + \frac{2}{L}x - 1)$
= $A(\cos 8.99\frac{x}{L} - 0.2227\sin 8.99\frac{x}{L} + \frac{2}{L}x - 1)$

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- Boundary Condition

$$w(0) = A + D = 0$$

$$w'(0) = \beta B + C = 0$$

$$M(L) = -EIw''(L) = -EI(-A\beta^{2} \cos\beta L - B\beta^{2} \sin\beta L) = 0$$

$$V(L) = -EI\frac{d^{3}w}{dx^{3}} - P\frac{dw}{dx} = 0$$

$$P = \frac{(2n-1)^{2}\pi^{2}EI}{4L^{2}}, \quad n = 1, 2, 3 \cdots$$

• Beam-Columns - Simple Beams with a Uniform Load



- Governing Equation

$$EI\frac{d^4w}{dx^4} + P\frac{d^2w}{dx^2} = q$$

- Particular Solution

$$w_p(x) = \frac{q}{2P} x^2$$

- General Solution

$$w(x) = A\cos\beta x + B\sin\beta x + Cx + D + \frac{q}{2P}x^{2}$$
$$w''(x) = -A\beta^{2}\cos\beta x - B\beta^{2}\sin\beta x + \frac{q}{P}$$

- Boundary Conditions

$$w(0) = A + D = 0 , \quad w''(0) = -EI(-A\beta^{2} + \frac{q}{P}) = 0 \to A = \frac{q}{P\beta^{2}}, D = -\frac{q}{P\beta^{2}}$$
$$w(L) = B\sin\beta L + CL + \frac{q}{P\beta^{2}}(\cos\beta L - 1) + \frac{q}{2P}L^{2} = 0$$
$$M(L) = -EIw''(L) = -EI(-A\beta^{2}\cos\beta L - B\beta^{2}\sin\beta L + \frac{q}{P}) = 0$$

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$$-\frac{q}{P}\cos\beta L - B\beta^{2}\sin\beta L + \frac{q}{P} = 0 \rightarrow B\beta^{2}\sin\beta L = -\frac{q}{P}(\cos\beta L - 1)$$
$$B\beta^{2}2\sin\frac{\beta L}{2}\cos\frac{\beta L}{2} = -\frac{q}{P}(1 - 2\sin^{2}\frac{\beta L}{2} - 1)$$
$$B = \frac{1}{\beta^{2}}\frac{q}{P}\tan\frac{\beta L}{2} , \quad C = -\frac{q}{2P}L$$

- Final Solution

$$w(x) = \frac{q}{\beta^2 P} (\cos\beta x - 1) + \frac{q}{\beta^2 P} \tan\frac{\beta L}{2} \sin\beta x - \frac{q}{2P} Lx + \frac{q}{2P} x^2$$
$$M(x) = -EIw''(x) = EI(\frac{q}{P} \cos\beta x + \frac{q}{P} \tan\frac{\beta L}{2} \sin\beta x - \frac{q}{P})$$

– Displacement at the mid-span

$$\begin{split} w(\frac{L}{2}) &= \frac{q}{\beta^2 P} (\cos \frac{\beta L}{2} - 1) + \frac{1}{\beta^2} \frac{q}{P} \tan \frac{\beta L}{2} \sin \frac{\beta L}{2} - \frac{q}{8P} L^2 \\ &= \frac{q}{\beta^2 P} (\cos \frac{\beta L}{2} + \tan \frac{\beta L}{2} \sin \frac{\beta L}{2} - 1) - \frac{q}{8P} L^2 \\ &= \frac{q}{\beta^2 P} (\frac{1}{\cos \frac{\beta L}{2}} - 1 - \frac{L^2 \beta^2}{8}) \\ &= \frac{5qL^4}{384EI} \frac{384}{5\pi^4} \frac{\pi^4 EI^2}{P^2 L^4} (\frac{1}{\cos \frac{\pi}{2} \sqrt{L^2 P / \pi^2 EI}} - 1 - \frac{\pi^2}{8} \frac{PL^2}{\pi^2 EI}) \\ &= \delta_{st} 0.7884 (\frac{P_{cr}}{P})^2 (\frac{1}{\cos \frac{\pi}{2} \sqrt{P / P_{cr}}} - 1 - \frac{\pi^2}{8} \frac{P}{P_{cr}}) \\ &= \delta_{st} \frac{0.7884}{t^2} (\frac{1}{\cos \frac{\pi}{2} \sqrt{t}} - 1 - \frac{\pi^2}{8} t) \quad (t = \frac{P}{P_{cr}}) \end{split}$$

– Moment at the Mid-span

$$M(\frac{L}{2}) = EI\frac{q}{P}(\cos\beta\frac{L}{2} + \tan\frac{\beta L}{2}\sin\frac{\beta L}{2} - 1)$$

= $\frac{qL^2}{8}\frac{8}{\pi^2}\frac{\pi^2 EI}{PL^2}(\cos\beta\frac{L}{2} + \tan\frac{\beta L}{2}\sin\frac{\beta L}{2} - 1)$
= $M_{st}\frac{8}{\pi^2}\frac{P_{cr}}{P}(\frac{1}{\cos\frac{\pi}{2}\sqrt{P/P_{cr}}} - 1)$
= $M_{st}0.811\frac{1}{t}(\frac{1}{\cos\frac{\pi}{2}\sqrt{t}} - 1)$

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Figure 2. Moment at the mid-span

3.3 Homogeneous and Particular solution

 $w = w_h + w_p = A\cos\beta x + B\sin\beta x + Cx + D + w_p$

$$EI\frac{d^{4}(w_{h}+w_{p})}{dx^{4}} + Q\frac{d^{2}(w_{h}+w_{p})}{dx^{2}} = EI\frac{d^{4}w_{h}}{dx^{4}} + P\frac{d^{2}w_{h}}{dx^{2}} + EI\frac{d^{4}w_{p}}{dx^{4}} + P\frac{d^{2}w_{p}}{dx^{2}}$$
$$= EI\frac{d^{4}w_{p}}{dx^{4}} + P\frac{d^{2}w_{p}}{dx^{2}} = q$$

• Four Boundary Conditions for Simple Beams

$$w(0) = A + D + w_{p}(0) = 0 , \quad M(0) = -EIw''(0) = -EI(-A\beta^{2} + w_{p}''(0)) = 0$$

$$w(L) = A\cos\beta L + B\sin\beta L + CL + D + w_{p}(L) = 0$$

$$M(L) = -EIw''(L) = -EI(-A\beta^{2}\cos\beta L - B\beta^{2}\sin\beta L + w_{p}''(L)) = 0$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ -\beta^2 & 0 & 0 & 0 \\ \cos\beta L & \sin\beta L & L & 1 \\ -\beta^2 \cos\beta L & -\beta^2 \sin\beta L & 0 & 0 \end{bmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} + \begin{pmatrix} w_p(0) \\ w''_p(0) \\ w_p(L) \\ w''_p(L) \end{pmatrix} = 0 \rightarrow \mathbf{K}\mathbf{X} + \mathbf{F} = 0$$

The homogenous solution is for the boundary conditions, while the particular solution is for the equilibrium.

3.4 Nonlinear equation solver

• Newton-Raphson Method (one point method)



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$$f(x) = 0 \to f(x_k) = f(x_{k-1} + \Delta x_k) = f(x_{k-1}) + \frac{df}{dx}\Big|_{x = x_{k-1}} \Delta x_k + \text{H.O.T.} \approx f(x_{k-1}) + \frac{df}{dx}\Big|_{x = x_{k-1}} \Delta x_k = 0$$

$$\Delta x_k = -\frac{f(x_{k-1})}{f'(x_{k-1})} \to x_k = x_{k-1} - \frac{f(x_{k-1})}{f'(x_{k-1})}$$

✓ Guarantees to exhibit the quadratic convergence rate

• Bisecton Method (two point method)



For $f(x_{k-1})f(x_k) < 0$, $\hat{f}_k(x) = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} x + \hat{f}_k(x_k) - \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} x_k$ $\hat{f}_k(x) = 0 \rightarrow x_{k+1} = x_k - \frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})} \hat{f}_k(x_k)$

✓ For the next iteration, select either $f(x_{k-1})f(x_{k+1}) < 0$ or $f(x_k)f(x_{k+1}) < 0$

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Chapter 4

Energy Principles

Principle of Minimum Potential Energy and Principle of Virtual Work





Read Chapter 11 (pp.420~ 428) of Elementary Structural Analysis 4th Edition by C .H. Norris *et al* very carefully. In this note an overbarred variable denotes a virtual quantity. The virtual displacement field should satisfy the displacement boundary conditions of supports if specified. For beam problems, displacement boundary conditions include boundary conditions for rotational angle. Variables with superscript *e* denote the exact solution that satisfies the equilibrium equation(s).

4.1 Spring-Force Systems



• Total Potential energy

The energy required to return a mechanical system to a reference status

$$\Pi_{int} = -\int_{\Delta}^{0} k(\Delta - u) du = \int_{0}^{\Delta} k(\Delta - u) du = \frac{1}{2} k\Delta^{2}, \quad \Pi_{ext} = -P\Delta$$
$$\Pi_{total} = \Pi_{int} + \Pi_{ext} = \frac{1}{2} k\Delta^{2} - P\Delta$$

- Equilibrium Equation $k\Delta^e = P$
- Principle of Minimum Potential Energy for an arbitrary displacement $\Delta = \Delta^e + \overline{\Delta}$.

$$\Pi_{total} = \frac{1}{2}k(\Delta^{e} + \overline{\Delta})^{2} - P(\Delta^{e} + \overline{\Delta})$$

$$= \frac{1}{2}k(\Delta^{e})^{2} + k\Delta^{e}\overline{\Delta} + \frac{1}{2}k(\overline{\Delta})^{2} - P(\Delta^{e} + \overline{\Delta})$$

$$= \frac{1}{2}k(\Delta^{e})^{2} - P\Delta^{e} + \frac{1}{2}k(\overline{\Delta})^{2} + \overline{\Delta}(k\Delta^{e} - P)$$

$$= \frac{1}{2}k(\Delta^{e})^{2} - P\Delta^{e} + \frac{1}{2}k(\overline{\Delta})^{2}$$

$$= \Pi^{e}_{total} + \frac{1}{2}k(\overline{\Delta})^{2} \ge \Pi^{e}_{total}$$

In the above equation, the equality sign holds if and only if $\overline{\Delta} = 0$. Therefore the total potential energy of the spring-force system becomes minimum when displacement of spring satisfies the equilibrium equation.

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4.2 Beam Problems



• Equilibrium Equation

$$EI\frac{d^2M^e}{dx^2} = -q \quad \text{or} \quad EI\frac{d^4w^e}{dx^4} = q$$

• Principle of Minimum Potential Energy for a virtual displacement $w = w^e + \overline{w}$.

$$\Pi^{h} = \frac{1}{2} \int_{0}^{l} \left(\frac{d^{2}(w^{e} + \overline{w})}{dx^{2}} EI \frac{d^{2}(w^{e} + \overline{w})}{dx^{2}} \right) dx - \int_{0}^{l} (w^{e} + \overline{w}) q dx$$

$$= \frac{1}{2} \int_{0}^{l} \left(\frac{d^{2}w^{e}}{dx^{2}} EI \frac{d^{2}w^{e}}{dx^{2}} \right) dx - \int_{0}^{l} w^{e} q dx +$$

$$\frac{1}{2} \int_{0}^{l} \left(\frac{d^{2}\overline{w}}{dx^{2}} EI \frac{d^{2}\overline{w}}{dx^{2}} \right) dx + \int_{0}^{l} \left(\frac{d^{2}\overline{w}}{dx^{2}} EI \frac{d^{2}w^{e}}{dx^{2}} \right) dx - \int_{0}^{l} \overline{w} q dx$$

$$= \Pi^{e} + \frac{1}{2} \int_{0}^{l} \left(\frac{d^{2}\overline{w}}{dx^{2}} EI \frac{d^{2}\overline{w}}{dx^{2}} \right) dx + \int_{0}^{l} \left(-EI \frac{d^{2}\overline{w}}{dx^{2}} \right) \frac{1}{EI} \left(-EI \frac{d^{2}w^{e}}{dx^{2}} \right) dx - \int_{0}^{l} \overline{w} q dx$$

$$= \Pi^{e} + \frac{1}{2} \int_{0}^{l} \left(\frac{d^{2}\overline{w}}{dx^{2}} EI \frac{d^{2}\overline{w}}{dx^{2}} \right) dx + \int_{0}^{l} \left(\frac{\overline{MM}^{e}}{EI} dx - \int_{0}^{l} \overline{w} q dx$$

$$= \Pi^{e} + \frac{1}{2} \int_{0}^{l} \left(\frac{d^{2}\overline{w}}{dx^{2}} EI \frac{d^{2}\overline{w}}{dx^{2}} \right) dx + \int_{0}^{l} \frac{\overline{MM}^{e}}{EI} dx - \int_{0}^{l} \overline{w} q dx$$

$$= \Pi^{e} + \frac{1}{2} \int_{0}^{l} \left(\frac{d^{2}\overline{w}}{dx^{2}} EI \frac{d^{2}\overline{w}}{dx^{2}} \right) dx + \int_{0}^{l} \frac{\overline{MM}^{e}}{EI} dx - \int_{0}^{l} \overline{w} q dx$$

Since the equation in the box represents the total virtual work in a beam, the total potential energy of a beam becomes minimum for all virtual displacement fields when the principle of virtual work holds. In the above equation, the equality sign holds if and only if $\overline{w} = 0$.

• Principle of Virtual Work

If a beam is in equilibrium, the principle of the virtual work holds for the beam,.

$$\int_{0}^{l} \overline{w} (EI \frac{d^{4} w^{e}}{dx^{4}} - q) dx = 0 \quad \text{for all virtual displacement} \quad \overline{w}$$

$$\int_{0}^{l} \frac{d^{2} \overline{w}}{dx^{2}} EI \frac{d^{2} w^{e}}{dx^{2}} dx - \int_{0}^{l} \overline{w} q dx + \frac{d\overline{w}}{dx} EI \frac{d^{2} w^{e}}{dx^{2}} \Big|_{0}^{l} - \overline{w} EI \frac{d^{3} w^{e}}{dx^{3}} \Big|_{0}^{l} = 0$$

$$\int_{0}^{l} \frac{d^{2} \overline{w}}{dx^{2}} EI \frac{d^{2} w^{e}}{dx^{2}} dx - \int_{0}^{l} \overline{w} q dx = \int_{0}^{l} \frac{\overline{M} M^{e}}{EI} dx - \int_{0}^{l} \overline{w} q dx = 0$$

In case that there is no support settlement, the boundary terms in above equation vanishes identically since either virtual displacement including virtual rotational angle or corresponding forces (moment and shear) vanish at supports. The principle of virtual work yields the displacement of an arbitrary point \tilde{x} in a beam by applying an unit load at \tilde{x} and by using the reciprocal theorem.

$$\int_{0}^{l} \overline{w} q dx = \int_{0}^{l} w \overline{q} dx = \int_{0}^{l} w \delta(x - \overline{x}) dx = w(\overline{x}) = \int_{0}^{l} \frac{\overline{M}M^{e}}{EI} dx$$

• Approximation using the principle of minimum potential energy

- Approximation of displacement field

$$w = \sum_{i=1}^{n} a_i g_i$$

- Total potential energy by the assumed displacement field

$$\Pi^{h} = \frac{1}{2} \int_{0}^{l} \left(\frac{d^{2}w}{dx^{2}} EI \frac{d^{2}w}{dx^{2}} \right) dx - \int_{0}^{l} wqdx = \frac{1}{2} \int_{0}^{l} \left(\sum_{i=1}^{n} a_{i} g_{i}'' \right) EI \left(\sum_{j=1}^{n} a_{j} g_{j}'' \right) dx - \int_{0}^{l} \sum_{i=1}^{n} a_{i} g_{i} qdx$$

- The first-order Necessary Condition

$$\frac{\partial \Pi^{h}}{\partial a_{k}} = \frac{\partial}{\partial a_{k}} \left(\frac{1}{2} \int_{0}^{l} \left(\sum_{i=1}^{n} a_{i} g_{i}''\right) EI\left(\sum_{i=1}^{n} a_{i} g_{i}''\right) dx - \int_{0}^{l} \sum_{i=1}^{n} a_{i} g_{i} q dx\right)$$
$$= \frac{1}{2} \left[\int_{0}^{l} g_{k}'' EI\left(\sum_{i=1}^{n} a_{j} g_{j}''\right) dx + \int_{0}^{l} \left(\sum_{i=1}^{n} a_{i} g_{i}''\right) EIg_{k}'' dx\right] - \int_{0}^{l} g_{k} q dx \quad \text{or} \quad \mathbf{Ka} = \mathbf{f}$$
$$= \sum_{i=1}^{n} \int_{0}^{l} g_{k}'' EIg_{i}'' dxa_{i} - \int_{0}^{l} g_{k} q dx = \sum_{i=1}^{n} K_{ki}a_{i} - f_{k} = 0$$

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• Example



i) with one unknown

$$w = ax(x-l) = a(x^{2} - lx) \rightarrow w'' = 2a$$

$$\Pi_{total} = \frac{1}{2} \int_{0}^{l} EI(w'')^{2} dx - \int_{0}^{l} P\delta(x - \frac{l}{2})w dx = \frac{1}{2} \int_{0}^{l} EI(2a)^{2} dx + aP \frac{l^{2}}{4} = \frac{1}{2} EI 4a^{2}l + aP \frac{l^{2}}{4}$$

$$\frac{\partial \Pi_{total}}{\partial a} = 0 \rightarrow 4aEIl + P \frac{l^{2}}{4} = 0 \rightarrow a = -\frac{Pl}{16EI} \rightarrow w = -\frac{Pl}{16EI}(x^{2} - xl)$$

$$w(\frac{l}{2}) = \frac{Pl^{3}}{64EI} \quad , \quad w^{e}(\frac{l}{2}) = \frac{Pl^{3}}{48EI} = 0.0208 \frac{Pl^{3}}{EI} \quad , \quad \text{Error} = \frac{w^{e}(\frac{l}{2}) - w(\frac{l}{2})}{w^{e}(\frac{l}{2})} = 0.25$$

ii) with two unknowns

$$w = ax(x-l) + bx(x^{2}-l^{2}) \rightarrow w'' = 2a + 6bx$$

$$\Pi_{total} = \frac{1}{2} \int_{0}^{l} EI(w'')^{2} dx - \int_{0}^{l} P\delta(x-\frac{l}{2})w dx = \frac{1}{2} \int_{0}^{l} EI(2a+6bx)^{2} dx - P(-a\frac{l^{2}}{4}-b\frac{3l^{3}}{8})$$

$$= \frac{1}{2} EI(4a^{2}l + 24ab\frac{l^{2}}{2} + 36b^{2}\frac{l^{3}}{3}) + P(a\frac{l^{2}}{4} + b\frac{3l^{3}}{8})$$

$$\frac{\partial \Pi_{total}}{\partial a} = 0 \rightarrow EI(4la+6l^{2}b) = -P\frac{l^{2}}{4}$$

$$\Rightarrow a = -\frac{1}{16}\frac{Pl}{EI}, \ b = 0 \quad (???)$$

iii) with three unknowns

$$w = ax(x-l) + bx(x^{2} - l^{2}) + cx(x^{3} - l^{3}) \rightarrow w'' = 2a + 6bx + 12cx^{2}$$

$$\Pi_{total} = \frac{1}{2} \int_{0}^{l} EI(w'')^{2} dx - \int_{0}^{l} P\delta(x - \frac{l}{2})w dx$$

$$= \frac{1}{2} \int_{0}^{l} EI(2a + 6bx + 12cx^{2})^{2} dx - P(-a\frac{l^{2}}{4} - b\frac{3l^{3}}{8} - c\frac{7l^{4}}{16})$$

$$= \frac{1}{2} EI(4a^{2}l + 36b^{2}\frac{l^{3}}{3} + 144c^{2}\frac{l^{5}}{5} + 24ab\frac{l^{2}}{2} + 48ac\frac{l^{3}}{3} + 144bc\frac{l^{4}}{4}) + P(a\frac{l^{2}}{4} + b\frac{3l^{3}}{8} + c\frac{7l^{4}}{16})$$

$$\frac{\partial \Pi_{total}}{\partial a} = 0 \to EI(4la + 6l^{2}b + 8l^{3}c) = -P\frac{l^{2}}{4}$$

$$\frac{\partial \Pi_{total}}{\partial b} = 0 \to EI(6l^{2}a + 12l^{3}b + 18l^{4}c) = -P\frac{3l^{3}}{8} \to \begin{cases} a = \frac{1}{64}\frac{Pl}{EI} \\ b = -\frac{5}{32}\frac{P}{EI} \\ c = \frac{5}{64}\frac{P}{EIl} \\ c = \frac{5}{64}\frac{P}{EIl} \end{cases}$$

$$w = \frac{1}{64} \frac{Pl}{EI} x(x-l) - \frac{5}{32} \frac{P}{EI} x(x^2 - l^2) + \frac{5}{64} \frac{P}{EII} x(x^3 - l^3)$$
$$w(\frac{l}{2}) = \frac{Pl^3}{EI} (-\frac{1}{64} \frac{1}{4} + \frac{5}{32} \frac{3}{8} - \frac{5}{64} \frac{7}{16}) = \frac{21}{1024} \frac{Pl^3}{EI} = 0.0205 \frac{Pl^3}{EI} , \text{ Error} = 0.0144$$

iv) with one sin function

$$w = a \sin \frac{\pi}{l} x \Longrightarrow w'' = a(\frac{\pi}{l})^2 \sin \frac{\pi}{l} x$$
$$\Pi_{total} = \frac{1}{2} \int_{0}^{l} EI(w'')^2 dx - \int_{0}^{l} P\delta(x - \frac{l}{2}) w dx = \frac{1}{2} \int_{0}^{l} EI(a(\frac{\pi}{l})^2 \sin \frac{\pi}{l} x)^2 dx - aP$$
$$= \frac{1}{2} EIa^2 (\frac{\pi}{l})^4 \int_{0}^{l} \sin^2 \frac{\pi}{l} x dx + aP = \frac{1}{2} EIa^2 (\frac{\pi}{l})^4 \frac{l}{2} + aP$$
$$\frac{\partial \Pi_{total}}{\partial a} = 0 \longrightarrow EIa(\frac{\pi}{l})^4 \frac{l}{2} - P = 0 \longrightarrow a = \frac{2}{\pi^4} \frac{Pl^3}{EI} \longrightarrow w = \frac{2}{\pi^4} \frac{Pl^3}{EI} \sin \frac{\pi}{l} x$$
$$w(\frac{l}{2}) = \frac{1}{48,7045} \frac{Pl^3}{EI} \quad , \quad w^e(\frac{l}{2}) = \frac{Pl^3}{48EI} \quad , \quad \text{Error} = 0.0145$$

v) with two sin function

$$w = a \sin \frac{\pi}{l} x + b \sin \frac{3\pi}{l} x \Longrightarrow w'' = a(\frac{\pi}{l})^2 \sin \frac{\pi}{l} x + b(\frac{3\pi}{l})^2 \sin \frac{3\pi}{l} x$$

$$\Pi_{total} = \frac{1}{2} \int_{0}^{l} EI(w'')^2 dx - \int_{0}^{l} P\delta(x - \frac{l}{2}) w dx$$

$$= \frac{1}{2} \int_{0}^{l} EI(a(\frac{\pi}{l})^2 \sin \frac{\pi}{l} x + b(\frac{3\pi}{l})^2 \sin \frac{3\pi}{l} x)^2 dx - aP + bP$$

$$= \frac{1}{2} EIa^2 (\frac{\pi}{l})^4 \int_{0}^{l} \sin^2 \frac{\pi}{l} x dx + EIab(\frac{\pi}{l})^2 (\frac{3\pi}{l})^2 \int_{0}^{l} \sin \frac{\pi}{l} \sin \frac{3\pi}{l} x dx + \frac{1}{2} EIb^2 (\frac{3\pi}{l})^4 \int_{0}^{l} \sin^2 \frac{3\pi}{l} x dx - aP + bP$$

$$= \frac{1}{2} EIa^2 (\frac{\pi}{l})^4 \int_{0}^{l} \sin^2 \frac{3\pi}{l} x dx - aP + bP$$

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$$\begin{aligned} \frac{\partial \Pi_{total}}{\partial a} &= 0 \to EIa(\frac{\pi}{l})^4 \frac{l}{2} - P = 0 \to a = \frac{2}{\pi^4} \frac{Pl^3}{EI} \\ \frac{\partial \Pi_{total}}{\partial b} &= 0 \to EIb(\frac{3\pi}{l})^4 \frac{l}{2} + P = 0 \to b = -\frac{2}{(3\pi)^4} \frac{Pl^3}{EI} \\ w &= \frac{2}{\pi^4} \frac{Pl^3}{EI} (\sin\frac{\pi}{l}x - \frac{1}{81}\sin\frac{3\pi}{l}x) \\ w(\frac{l}{2}) &= 0.0205(1 + 0.0123) \frac{Pl^3}{EI} = 0.0208 \frac{Pl^3}{EI} , \ w^e(\frac{l}{2}) = 0.0208 \frac{Pl^3}{EI} , \text{ Error } \cong 0 \end{aligned}$$

4.3 Truss problems



• Potential Energy

$$\Pi_{int} = \frac{1}{2} \sum_{i=1}^{nmb} \frac{(F_i)^2 l_i}{EA^i} , \quad \Pi_{ext} = -\sum_{i=1}^{njn} (X_i u_i + Y_i v_i)$$
$$\Pi_{total} = \Pi_{int} + \Pi_{ext} = \frac{1}{2} \sum_{i=1}^{nmb} \frac{(F_i)^2 l_i}{EA^i} - \sum_{i=1}^{njn} (X_i u_i + Y_i v_i)$$

where *nmb* and *njn* denotes the total number of members and the total numbers of joints in a truss.

• Equilibrium Equations

$$-\sum_{j=1}^{m(i)} H_{j}^{i} + X^{i} = 0 , -\sum_{j=1}^{m(i)} V_{j}^{i} + Y^{i} = 0 \text{ for } i = 1, \dots, njn$$

where m(i), H_j^i and V_j^i are the number of member connected to joint *i*, the horizontal component and the vertical component of the bar force of *j*-th member connected to joint *i*, respectively.

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• Principle of Minimum Potential Energy

$$\begin{aligned} \Pi_{total} &= \frac{1}{2} \sum_{i=1}^{nmb} \frac{(F_i^e + F_i^-)^2 l_i}{EA_i} - \sum_{i=1}^{njn} (X_i (u_i^- + \overline{u}_i^-) + Y_i (v_i^- + \overline{v}_i^-)) \\ &= \frac{1}{2} \sum_{i=1}^{nmb} \frac{(F_i^e)^2 l_i}{EA_i} - \sum_{i=1}^{njn} (X_i u_i^- + Y_i v_i^-) + \frac{1}{2} \sum_{i=1}^{nmb} \frac{(\overline{F}_i^-)^2 l_i}{EA_i^-} + \sum_{i=1}^{nmb} \frac{F_i^e \overline{F}_i^- l_i^-}{EA_i^-} - \sum_{i=1}^{njn} (X_i \overline{u}_i^- + Y_i \overline{v}_i^-) \\ &= \frac{1}{2} \sum_{i=1}^{nmb} \frac{(F_i^e)^2 l_i^-}{EA_i^-} - \sum_{i=1}^{njn} (X_i u_i^- + Y_i v_i^-) + \sum_{i=1}^{nmb} \frac{(\overline{F}_i^-)^2 l_i^-}{EA_i^-} \\ &= \Pi^e + \frac{1}{2} \sum_{i=1}^{nmb} \frac{(\overline{F}_i^-)^2 l_i^-}{EA_i^-} \ge \Pi^e \quad \text{for all virtual displacement fields} \end{aligned}$$

where F_i^e and $\overline{F_i}$ are the bar force of *i*-th member induced by the real displacement of joints and virtual displacement induced by the virtual displacement of joints. Since the equation in the box represents the total virtual work in a truss, the total potential energy of a truss becomes minimum for all virtual displacement fields when the principle of virtual work holds. In the above equation, the equality sign holds if and only if the virtual displacements at all joints are zero.

• Virtual Work Expression

If a truss is in equilibrium, the principle of the virtual work holds for the truss,.

$$\sum_{i=1}^{njn} ((-\sum_{j=1}^{m(i)} H_j^i + X^i) \overline{u}^i + (-\sum_{j=1}^{m(i)} V_j^i + Y^i) \overline{v}^i) = 0$$



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$$\begin{split} &\sum_{i=1}^{njn} \left(\left(-\sum_{j=1}^{m(i)} F_j^i \cos \theta_j + X_i \right) \overline{u}_i + \left(-\sum_{j=1}^{m(i)} F_j^i \sin \theta_j + Y_i \right) \overline{v}_i \right) = 0 \\ &\sum_{i=1}^{njn} \left(\overline{u}^i \sum_{j=1}^{m(i)} F_j^i \cos \theta_j + \overline{v}^i \sum_{j=1}^{m(i)} F_j^i \sin \theta_j \right) = \sum_{i=1}^{njn} \left(X_i \overline{u}_i + Y_i \overline{v}_i \right) \\ &\sum_{i=1}^{nmb} \left(F_i^e \cos \theta_i (\overline{u}_i^2 - \overline{u}_i^1) + F_i^e \sin \theta_i (\overline{v}_i^2 - \overline{v}_i^1) \right) = \sum_{i=1}^{njn} \left(X_i \overline{u}_i + Y_i \overline{v}_i \right) \\ &\sum_{i=1}^{nmb} F_i^e \left(\cos \theta_i (\overline{u}_i^2 - \overline{u}_i^1) + \sin \theta_i (\overline{v}_i^2 - \overline{v}_i^1) \right) = \sum_{i=1}^{nmb} F_i^e \Delta \overline{l}_i = \sum_{i=1}^{nmb} F_i^e \frac{\overline{F}_i \ l_i}{(EA_i)} = \sum_{i=1}^{nmb} \frac{F_i^e \overline{F}_i \ l_i}{(EA_i)} \end{split}$$

The principle of virtual work yields the displacement of a joint k in a truss by applying an unit load at a joint k in an arbitrary direction and by using the reciprocal theorem.

$$\overline{X}_k u_k + \overline{Y}_k v_k = \|\mathbf{X}\| \|\mathbf{u}_k\| \cos \alpha = \|\mathbf{u}_k\| \cos \alpha = \sum_{i=1}^{nmb} \frac{F_i^e \overline{F}_i l_i}{(EA_i)}$$

Since α represents the angle between the applied unit load and the displacement vector, $\|\mathbf{u}_k\|\cos\alpha$ are the displacement of the joint *k* in the direction of the applied unit load.

4.4. Buckling Problems

• Total Potential Energy



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• Principle of the Minimum Potential Energy

$$\Pi^{h} = \frac{1}{2} \int_{0}^{l} \frac{d^{2}(w^{e} + \overline{w})}{dx^{2}} EI \frac{d^{2}(w^{e} + \overline{w})}{dx^{2}} dx - \frac{1}{2} \int_{0}^{l} \frac{d(w^{e} + \overline{w})}{dx} P \frac{d(w^{e} + \overline{w})}{dx} dx - \int_{0}^{l} (w^{e} + \overline{w}) q dx$$

$$= \frac{1}{2} \int_{0}^{l} (\frac{d^{2}w^{e}}{dx^{2}} EI \frac{d^{2}w^{e}}{dx^{2}} - \frac{dw^{e}}{dx} P \frac{dw^{e}}{dx}) dx - \int_{0}^{l} w^{e} q dx + \int_{0}^{l} (\frac{d^{2}w^{e}}{dx^{2}} EI \frac{d^{2}\overline{w}}{dx^{2}} - \frac{dw^{e}}{dx} P \frac{d\overline{w}}{dx}) dx$$

$$- \int_{0}^{l} \overline{w} q dx + \frac{1}{2} \int_{0}^{l} (\frac{d^{2}\overline{w}}{dx^{2}} EI \frac{d^{2}\overline{w}}{dx^{2}} - \frac{d\overline{w}}{dx} P \frac{d\overline{w}}{dx}) dx$$

$$= \Pi^{e} + \int_{0}^{l} \overline{w} (EI \frac{d^{4}w^{e}}{dx^{4}} + P \frac{d^{2}w^{e}}{dx^{2}} - q) dx + \frac{1}{2} \int_{0}^{l} (\frac{d^{2}\overline{w}}{dx^{2}} EI \frac{d^{2}\overline{w}}{dx} P \frac{d\overline{w}}{dx}) dx$$

$$= \Pi^{e} + \frac{1}{2} \int_{0}^{l} (\frac{d^{2}\overline{w}}{dx^{2}} EI \frac{d^{2}\overline{w}}{dx^{2}} - \frac{d\overline{w}}{dx} P \frac{d\overline{w}}{dx}) dx$$

• The principle of minimum potential energy holds if and only if

$$\int_{0}^{l} \left(\frac{d^{2}\overline{w}}{dx^{2}} EI \frac{d^{2}\overline{w}}{dx^{2}} - \frac{d\overline{w}}{dx} P \frac{d\overline{w}}{dx}\right) dx > 0 \text{ for all possible } \overline{w}$$

• The principle of the minimum potential energy is not valid for the following cases.

$$\int_{0}^{l} \left(\frac{d^{2}\overline{w}}{dx^{2}} EI \frac{d^{2}\overline{w}}{dx^{2}} - \frac{d\overline{w}}{dx} P \frac{d\overline{w}}{dx}\right) dx \le 0 \text{ for some } \overline{w}$$

• The critical status of a structure is defined as

$$\int_{0}^{l} \left(\frac{d^{2}\overline{w}}{dx^{2}} EI \frac{d^{2}\overline{w}}{dx^{2}} - \frac{d\overline{w}}{dx} P \frac{d\overline{w}}{dx}\right) dx = 0$$

• Approximation

- Approximation of displacement: $\overline{w} = \sum_{i=1}^{n} a_i g_i$

- Critical Status

$$\int_{0}^{l} (\sum_{i=1}^{n} a_{i} g_{i}'') EI(\sum_{j=1}^{n} a_{j} g_{j}'') dx - P \int_{0}^{l} (\sum_{i=1}^{n} a_{i} g_{i}') (\sum_{j=1}^{n} a_{j} g_{j}') dx =$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{i} \int_{0}^{l} g_{i}'' EIg_{i}'' dx a_{j} - \sum_{i=1}^{n} \sum_{j=1}^{n} a_{i} P \int_{0}^{l} g_{i}' g_{j}' dx a_{j} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{i} K_{ij} a_{j} - \sum_{i=1}^{n} \sum_{j=1}^{n} a_{i} K_{ij}^{G} a_{j} =$$

$$(a)^{T} (\mathbf{K} - \mathbf{K}^{G})(a)^{T} = 0 \rightarrow Det(\mathbf{K} - \mathbf{K}^{G}) = 0$$

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• Example - Simple Beam



- with a parabola:
$$w = ax(x-l) \rightarrow g'_1 = 2x-l$$
, $g''_1 = 2$

$$\int_{0}^{l} g_{i}^{"}EIg_{i}^{"}dx = \int_{0}^{l} 2EI2dx = 4EIl$$

$$\int_{0}^{l} g_{i}^{'}EIg_{i}^{'}dx = \int_{0}^{l} (2x-l)^{2} dx = \int_{0}^{l} (4x^{2} - 4xl + l^{2}) dx = (\frac{4}{3} - 2 + 1)l^{3} = \frac{1}{3}l^{3}$$

$$Det(4EIl - P\frac{1}{3}l^{3}) = 0 \rightarrow P_{cr} = \frac{12EI}{l^{2}} \quad (exact : \frac{\pi^{2}EI}{l^{2}} = 9.86\frac{EI}{l^{2}}, error = 22\%)$$

- with one sine curve: $w = a \sin \frac{\pi x}{l} \rightarrow g'_1 = \frac{\pi}{l} \cos \frac{\pi x}{l}, g''_1 = (\frac{\pi}{l})^2 \sin \frac{\pi x}{l}$

$$\int_{0}^{l} g_{i}''EIg_{i}''dx = EI(\frac{\pi}{l})^{4} \int_{0}^{l} \sin^{2} \frac{\pi x}{l} dx = EI(\frac{\pi}{l})^{4} \frac{l}{2}$$
$$\int_{0}^{l} g_{i}'EIg_{i}'dx = (\frac{\pi}{l})^{2} \int_{0}^{l} \cos^{2} \frac{\pi x}{l} dx = (\frac{\pi}{l})^{2} \frac{l}{2}$$
$$Det(EI(\frac{\pi}{l})^{4} \frac{l}{2} - P(\frac{\pi}{l})^{2} \frac{l}{2}) = 0 \rightarrow P_{cr} = \frac{\pi^{2}EI}{l^{2}} (\text{exact})$$

• Example – Cantilever Beam

$$P \longrightarrow P$$

– with one unknown:

$$w = ax^{2} \rightarrow g_{1}' = 2x, \ g_{1}'' = 2$$

$$\int_{0}^{l} g_{i}''EIg_{i}''dx = \int_{0}^{l} 2EI2dx = 4EIl, \ \int_{0}^{l} g_{i}'EIg_{i}'dx = \int_{0}^{l} 4x^{2}dx = \frac{4}{3}l^{3}$$

$$Det(4EIl - P\frac{4}{3}l^{3}) = 0 \rightarrow P_{cr} = \frac{3EI}{l^{2}} \quad (exact: \frac{\pi^{2}EI}{4l^{2}} = 2.46\frac{EI}{l^{2}}, \ error = 22\%)$$
- with two unknowns:

$$w = ax^{2} + bx^{3} \rightarrow g_{1}' = 2x, g_{2}' = 3x^{2}, g_{1}'' = 2, g_{2}'' = 6x$$

$$K_{11}^{G} = P_{0}^{l} 4x^{2} dx = \frac{4}{3} Pl^{3}, K_{12}^{G} = K_{21}^{G} = P_{0}^{l} 6x^{3} dx = \frac{6}{4} Pl^{4}, K_{22}^{G} = P_{0}^{l} 9x^{4} dx = \frac{9}{5} Pl^{5}$$

$$K_{11} = \int_{0}^{l} 4dx = 4l, K_{12} = K_{21} = \int_{0}^{l} 12x dx = 6l^{2}, K_{22} = \int_{0}^{l} 36x^{2} dx = 12l^{3}$$

$$Det(EI \begin{bmatrix} 4l & 6l^{2} \\ 6l^{2} & 12l^{3} \end{bmatrix} - P \frac{1}{30} \begin{bmatrix} 40l^{3} & 45l^{4} \\ 45l^{4} & 54l^{5} \end{bmatrix}) = 0 \rightarrow \begin{vmatrix} 4l - 40l^{3}\alpha & 6l^{2} - 45l^{4}\alpha \\ 6l^{2} - 45l^{4}\alpha & 12l^{3} - 54l^{5}\alpha \end{vmatrix} = 0, \alpha = \frac{1}{30} \frac{P}{EI}$$

$$(4l - 40l^{3}\alpha)(12l^{3} - 54l^{5}\alpha) - (6l^{2} - 45l^{4}\alpha)^{2} = 0 \rightarrow 4l^{4} - 52l^{6}\alpha + 45l^{8}\alpha^{2} = 0$$

$$\alpha = \frac{26l^6 \pm \sqrt{26^2 l^{12} - 180l^{12}}}{45l^8} = \frac{26 \pm 22.27}{45l^2} = \frac{0.0829}{l^2} \text{ or } \frac{1.0727}{l^2} \rightarrow P_{cr} = 2.487 \frac{EI}{l^2} \text{ or } 32.181 \frac{EI}{l^2}$$
$$P_{exact} = 2.49 \frac{EI}{l^2} (\text{error} = 1.2\%) \text{ or } P_{exact} = 22.19 \frac{EI}{l^2} (\text{error} = 45\%)$$

• Beam-Columns - Simple Beams with a Uniform Load



- with one sine curve:

$$w = a \sin \frac{\pi x}{l} \to g_{1}' = \frac{\pi}{l} \cos \frac{\pi x}{l}, \ g_{1}'' = (\frac{\pi}{l})^{2} \sin \frac{\pi x}{l}$$

$$\int_{0}^{l} g_{i}'' E I g_{i}'' dx = E I(\frac{\pi}{l})^{4} \int_{0}^{l} \sin^{2} \frac{\pi x}{l} dx = E I(\frac{\pi}{l})^{4} \frac{l}{2}, \ \int_{0}^{l} g_{i}' g_{i}' dx = (\frac{\pi}{l})^{2} \int_{0}^{l} \cos^{2} \frac{\pi x}{l} dx = (\frac{\pi}{l})^{2} \frac{l}{2}$$

$$\int_{0}^{l} g_{i} q dx = q \int_{0}^{l} \sin \frac{\pi x}{l} dx = 2q \frac{l}{\pi}$$

$$\operatorname{Min} \Pi = \frac{1}{2} E I(\frac{\pi}{l})^{4} \frac{l}{2} a^{2} - \frac{1}{2} P(\frac{\pi}{l})^{2} \frac{l}{2} a^{2} - 2q \frac{l}{\pi} a \to$$

$$\frac{\partial \Pi}{\partial a} = E I(\frac{\pi}{l})^{4} \frac{l}{2} a - P(\frac{\pi}{l})^{2} \frac{l}{2} a - 2q \frac{l}{\pi} = 0$$

$$a = \frac{4q/\pi}{E I(\frac{\pi}{l})^{4} - P(\frac{\pi}{l})^{2}} = \frac{4q}{\pi} \frac{1}{E I} (\frac{l}{\pi})^{4} \frac{4q/\pi}{1 - \frac{P}{E I} (\frac{l}{\pi})^{2}} = \frac{5q l^{4}}{384E I} \frac{384 \times 4}{5\pi^{5}} \frac{1}{1 - P/P_{cr}} = \delta_{st} \frac{1.0038}{1 - t}$$

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Chapter 5

Matrix Structural Analysis



Mr. Force & Ms. Displacement Matchmaker: Stiffness Matrix

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5.1 Truss Problems

5.1.1 Member Stiffness Matrix



• Force – Displacement relation at Member ends

$$f_x^L = -\frac{EA}{L} (\delta_x^R - \delta_x^L)$$
$$f_x^R = \frac{EA}{L} (\delta_x^R - \delta_x^L)$$
$$f_y^L = f_y^R = 0$$

• Member Stiffness Matrix in Local Coordinate System

$$\begin{pmatrix} f_x^L \\ f_y^L \\ f_x^R \\ f_x^R \\ f_y^R \end{pmatrix}^e = \frac{EA}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \delta_x^L \\ \delta_y^R \\ \delta_x^R \\ \delta_y^R \end{pmatrix}^e$$
$$(\mathbf{f})^e = [\mathbf{k}]^e (\mathbf{\delta})^e$$

• Transformation Matrix



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• Member End Force

$$\begin{pmatrix} F_x^1 \\ F_y^1 \\ F_x^2 \\ F_y^2 \end{pmatrix}^e = \begin{bmatrix} \cos\phi & -\sin\phi & 0 & 0 \\ \sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & \cos\phi & -\sin\phi \\ 0 & 0 & \sin\phi & \cos\phi \end{bmatrix} \begin{pmatrix} f_x^L \\ f_y^L \\ f_x^R \\ f_y^R \end{pmatrix}^e \rightarrow \underline{(\mathbf{F})^e = [\Gamma]^T (\mathbf{f})^e}$$

• Member End Displacement

$$\begin{pmatrix} \delta_x^L \\ \delta_y^R \\ \delta_x^R \\ \delta_y^R \end{pmatrix}^e = \begin{bmatrix} \cos\phi & \sin\phi & 0 & 0 \\ -\sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & \cos\phi & \sin\phi \\ 0 & 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{pmatrix} \Delta_x^1 \\ \Delta_y^1 \\ \Delta_x^2 \\ \Delta_y^2 \end{pmatrix}^e \rightarrow \underbrace{(\delta)^e = [\Gamma](\Delta)^e}_{=}$$

• Member Stiffness Matrix in Global Coordinate

$$(\mathbf{F})^{e} = [\Gamma]^{T}(\mathbf{f})^{e} = [\Gamma]^{T}[\mathbf{k}]^{e}(\boldsymbol{\delta})^{e} = [\Gamma]^{T}[\mathbf{k}]^{e}[\Gamma](\boldsymbol{\Delta})^{e}$$

$$(\mathbf{F})^e = [\mathbf{K}]^e (\Delta)^e$$

$$[\mathbf{K}]^{e} = \frac{EA}{L} \begin{bmatrix} \cos^{2}\phi & \sin\theta\cos\phi & -\cos^{2}\phi & -\sin\phi\cos\phi \\ \sin\theta\cos\phi & \sin^{2}\phi & -\sin\phi\cos\phi & -\sin^{2}\phi \\ -\cos^{2}\phi & -\sin\theta\cos\phi & \cos^{2}\phi & \sin\phi\cos\phi \\ -\sin\phi\cos\phi & -\sin^{2}\phi & \sin\phi\cos\phi & \sin^{2}\phi \end{bmatrix} = \begin{bmatrix} [\mathbf{K}_{11}]^{e} & [\mathbf{K}_{12}]^{e} \\ [\mathbf{K}_{21}]^{e} & [\mathbf{K}_{22}]^{e} \end{bmatrix}$$

5.1.2 Global Stiffness Equation







Contragradient

$$(\mathbf{P})^{T} \cdot (\mathbf{u}) = (\mathbf{F})^{T} \cdot (\Delta) \to (\mathbf{P})^{T} \cdot (\mathbf{u}) = (\mathbf{F})^{T} [\mathbf{C}](\mathbf{u}) \to$$
$$((\mathbf{P})^{T} - (\mathbf{F})^{T} [\mathbf{C}])(\mathbf{u}) = 0 \text{ for all possible} \quad (\mathbf{u}) \to (\mathbf{P})^{T} = (\mathbf{F})^{T} [\mathbf{C}]$$
$$(\mathbf{P}) = [\mathbf{C}]^{T} (\mathbf{F}) = [\mathbf{E}](\mathbf{F}) \to [\mathbf{C}]^{T} = [\mathbf{E}]$$

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• Unassembled Member Stiffness Equation

$$\begin{pmatrix} (\mathbf{F})^{1} \\ \vdots \\ (\mathbf{F})^{i} \\ \vdots \\ (\mathbf{F})^{p} \end{pmatrix} = \begin{bmatrix} [\mathbf{K}]^{1} & \cdots & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & \cdots & [\mathbf{K}]^{i} & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & \cdots & [\mathbf{K}]^{p} \end{bmatrix} \begin{pmatrix} (\Delta)^{1} \\ \vdots \\ (\Delta)^{i} \\ \vdots \\ (\Delta)^{p} \end{pmatrix} \rightarrow \underline{(\mathbf{F}) = [\overline{\mathbf{K}}](\Delta)}$$

• Global Stiffness Equation

$$(\mathbf{P}) = [\mathbf{C}]^{T}(\mathbf{F}) = [\mathbf{C}]^{T}[\overline{\mathbf{K}}](\Delta) = [\mathbf{C}]^{T}[\overline{\mathbf{K}}][\mathbf{C}](\mathbf{u})$$
$$\underline{(\mathbf{P}) = [\mathbf{K}](\mathbf{u})} \quad \text{where} \quad \underline{[\mathbf{K}] = [\mathbf{C}]^{T}[\overline{\mathbf{K}}][\mathbf{C}]}$$

• Direct Stiffness Method

$$[\mathbf{K}] = [\mathbf{C}]^{T} [\overline{\mathbf{K}}] [\mathbf{C}] = \left[[\mathbf{C}]^{1^{T}} \cdots [\mathbf{C}]^{p^{T}} \right]^{T} \begin{bmatrix} [\mathbf{K}]^{1} & \cdots & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & \cdots & [\mathbf{K}]^{i} & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & \cdots & [\mathbf{K}]^{p} \end{bmatrix} \begin{bmatrix} [\mathbf{C}]^{1} \\ \vdots \\ [\mathbf{C}]^{i} \\ \vdots \\ [\mathbf{C}]^{p} \end{bmatrix}$$

$$= [\mathbf{C}]^{1'} [\mathbf{K}]^{1} [\mathbf{C}]^{1} + \dots + [\mathbf{C}]^{i'} [\mathbf{K}]^{i} [\mathbf{C}]^{i} + \dots + [\mathbf{C}]^{p'} [\mathbf{K}]^{p} [\mathbf{C}]^{p}$$

$$[\mathbf{C}]^{i^{T}} [\mathbf{K}]^{i} [\mathbf{C}]^{i} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \vdots & \vdots \\ \mathbf{I} & \mathbf{0} \\ \vdots & \mathbf{I} \\ [\mathbf{K}_{21}]^{i} & [\mathbf{K}_{22}]^{i} \end{bmatrix} \begin{bmatrix} \mathbf{0} & \cdots & \mathbf{I} & \cdots & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{I} & \cdots & \mathbf{0} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \vdots & \vdots \\ \mathbf{I} & \mathbf{0} \\ \vdots & \vdots \\ \mathbf{0} & \mathbf{I} \\ \vdots & \vdots \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{0} & \cdots & [\mathbf{K}_{11}]^{i} & \cdots & [\mathbf{K}_{12}]^{i} & \cdots & \mathbf{I} \\ \mathbf{0} & \cdots & [\mathbf{K}_{21}]^{i} & \cdots & [\mathbf{K}_{22}]^{i} & \cdots & \mathbf{0} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \vdots & \vdots \\ \mathbf{0} & \mathbf{I} \\ \vdots & \vdots \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{0} & \cdots & [\mathbf{K}_{11}]^{i} & \cdots & [\mathbf{K}_{12}]^{i} & \cdots & \mathbf{0} \\ \mathbf{0} & \cdots & [\mathbf{K}_{21}]^{i} & \cdots & [\mathbf{K}_{22}]^{i} & \cdots & \mathbf{0} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

5.1.3 Example



• Member Stiffness Matrix

$$\left[\mathbf{K}\right]^{e} = \frac{EA}{L} \begin{bmatrix} \cos^{2}\phi & \sin\phi\cos\phi & -\cos^{2}\phi & -\sin\phi\cos\phi \\ \sin\phi\cos\phi & \sin^{2}\phi & -\sin\phi\cos\phi & -\sin^{2}\phi \\ -\cos^{2}\phi & -\sin\phi\cos\phi & \cos^{2}\phi & \sin\phi\cos\phi \\ -\sin\phi\cos\phi & -\sin^{2}\phi & \sin\phi\cos\phi & \sin^{2}\phi \end{bmatrix} = \begin{bmatrix} \left[\mathbf{K}_{11}\right]^{e} & \left[\mathbf{K}_{12}\right]^{e} \\ \left[\mathbf{K}_{21}\right]^{e} & \left[\mathbf{K}_{22}\right]^{e} \end{bmatrix}$$

- Member 1: $\phi = 45^{\circ}$

$$\left[\mathbf{K}\right]^{1} = \frac{EA}{\sqrt{2}L} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

- Member 2: $\phi = -45^{\circ}$

$$\left[\mathbf{K}\right]^{2} = \frac{EA}{\sqrt{2}L} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

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- Member 3: $\phi = 180^{\circ}$

$$\left[\mathbf{K}\right]^{3} = \frac{EA}{2L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$



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• Global Stiffness Matrix

$$[\mathbf{K}] = [\mathbf{C}]^{T} [\overline{\mathbf{K}}] [\mathbf{C}] = [\mathbf{C}]^{T} [\mathbf{K}]^{1} [\mathbf{C}]^{1} + [\mathbf{C}]^{2T} [\mathbf{K}]^{2} [\mathbf{C}]^{2} + \dots + [\mathbf{C}]^{3T} [\mathbf{K}]^{3} [\mathbf{C}]^{3}$$

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• Stiffness Equation

$$\begin{array}{c} \begin{array}{c} \text{Unknown} & P_{1} \\ \text{Unknown} & P_{2} \\ \text{Known} & P_{3} \\ \text{Known} & P_{4} \\ \text{Known} & P_{5} \\ \text{Unknown} & P_{6} \end{array} = \frac{EA}{L} \\ = \frac{EA}{L} \left[\begin{array}{cccc} \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & 0 & 0 \\ -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & 0 & -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & 0 & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & 0 & -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & 0 & -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\ 0 & 0 & \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\ 0 & 0 & \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\ \end{array} \right] \end{array}$$

• Application of Support Conditions (Boundary Conditions)

$$\begin{pmatrix} P_{1} \\ P_{2} \\ P_{3} \\ P_{4} \\ P_{5} \\ P_{6} \end{pmatrix} = \frac{EA}{L} \begin{bmatrix} \frac{1+\sqrt{2}}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2} \\ -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & 0 & 0 \\ -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & 0 & -\frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & 0 & -\frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1+\sqrt{2}}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\ 0 & 0 & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\ 0 & 0 & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\ 0 & 0 & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\ 0 & 0 & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\ 0 & 0 & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\ 0 & 0 & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\ 0 & 0 & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\ 0 & 0 & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\ 0 & 0 & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\ 0 & 0 & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\ 0 & 0 & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\ 0 & 0 & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\ 0 & 0 & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\ 0 & 0 & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt$$

• Final Stiffness Equation

$$\begin{pmatrix} P_{3} \\ P_{4} \\ P_{5} \end{pmatrix} = \frac{EA}{L} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{2\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1+\sqrt{2}}{2\sqrt{2}} \end{bmatrix}^{(u_{3})}_{(u_{4})} \rightarrow \begin{pmatrix} u_{3} \\ u_{4} \\ u_{5} \end{pmatrix} = \frac{L}{EA} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{2\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1+\sqrt{2}}{2\sqrt{2}} \end{bmatrix}^{-1}_{(u_{3})}_{(u_{4})}$$

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5.2 Beam Problems

5.2.1 Member Stiffness Matrix



• Force-Displacement Relation at Member Ends

$$\begin{split} m^{L} &= \frac{4EI_{e}}{L_{e}} \Theta^{L} + \frac{2EI_{e}}{L_{e}} \Theta^{R} + \frac{6EI_{e}}{L_{e}^{2}} \delta_{y}^{L} - \frac{6EI_{e}}{L_{e}^{2}} \delta_{y}^{R} \\ m^{R} &= \frac{2EI_{e}}{L_{e}} \Theta^{L} + \frac{4EI_{e}}{L_{e}} \Theta^{R} + \frac{6EI_{e}}{L_{e}^{2}} \delta_{y}^{L} - \frac{6EI_{e}}{L_{e}^{2}} \delta_{y}^{R} \\ f_{y}^{L} &= \frac{M_{e}^{L} + M_{e}^{R}}{L_{e}} = \frac{6EI_{e}}{L_{e}^{2}} \Theta^{L} + \frac{6EI_{e}}{L_{e}^{2}} \Theta^{R} + \frac{12EI_{e}}{L_{e}^{3}} \delta_{y}^{L} - \frac{12EI_{e}}{L_{e}^{3}} \delta_{y}^{R} \\ f_{y}^{R} &= -\frac{M_{e}^{L} + M_{e}^{R}}{L_{e}} = -\frac{6EI_{e}}{L_{e}^{2}} \Theta^{L} - \frac{6EI_{e}}{L_{e}^{2}} \Theta^{R} - \frac{12EI_{e}}{L_{e}^{3}} \delta_{y}^{L} + \frac{12EI_{e}}{L_{e}^{3}} \delta_{y}^{R} \end{split}$$

• Transformation Matrix is not required

$$f \to F$$
, $m \to M$, $\delta \to \Delta$, $\theta \to \Theta$

• Member Stiffness Matrix

$$\begin{pmatrix} F_{y}^{1} \\ M^{1} \\ F_{y}^{2} \\ M^{2} \end{pmatrix}^{e} = \frac{EI_{e}}{L_{e}} \begin{bmatrix} \frac{12}{L_{e}^{2}} & \frac{6}{L_{e}} & -\frac{12}{L_{e}^{2}} & \frac{6}{L_{e}} \\ \frac{6}{L_{e}} & 4 & -\frac{6}{L_{e}} & 2 \\ -\frac{12}{L_{e}^{2}} & -\frac{6}{L_{e}} & \frac{12}{L_{e}^{2}} & -\frac{6}{L_{e}} \\ \frac{6}{L_{e}} & 2 & -\frac{6}{L_{e}} & 4 \end{bmatrix} \begin{pmatrix} \Delta^{1}_{y} \\ \Theta^{1} \\ \Delta^{2}_{y} \\ \Theta^{2}_{y} \end{pmatrix}^{e} \text{ or } \underline{(\mathbf{F})^{e} = [\mathbf{K}]^{e} (\Delta)^{e}}$$

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5.2.2 Global Stiffness Matrix



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$$(\mathbf{\Delta})_{i} = \begin{pmatrix} (\mathbf{\Delta})_{i}^{L} \\ (\mathbf{\Delta})_{i}^{R} \end{pmatrix} = \begin{bmatrix} 0 & \Box & \mathbf{I} & 0 & \Box & 0 \\ 0 & \Box & 0 & \mathbf{I} & \Box & 0 \end{bmatrix} \begin{pmatrix} \mathbf{u}^{1} \\ \Box \\ \mathbf{u}^{i} \\ \mathbf{u}^{i+1} \\ \Box \\ \mathbf{u}^{i+1} \\ \Box \\ \mathbf{u}^{p+1} \end{pmatrix} = [\mathbf{C}]_{i}(\mathbf{u})$$
$$(\mathbf{\Delta}) = \begin{pmatrix} (\mathbf{\Delta})_{1} \\ \vdots \\ (\mathbf{\Delta})_{p} \end{pmatrix} = \begin{bmatrix} [\mathbf{C}]_{1} \\ \vdots \\ [\mathbf{C}]_{p} \end{bmatrix} \begin{pmatrix} \mathbf{u}^{1} \\ \vdots \\ \mathbf{u}^{p+1} \end{pmatrix} = [\mathbf{C}](\mathbf{u})$$

• Unassembled Member Stiffness Equation

$$\begin{pmatrix} (\mathbf{F})_{1} \\ \Box \\ (\mathbf{F})_{i} \\ \Box \\ (\mathbf{F})_{p} \end{pmatrix} = \begin{bmatrix} [\mathbf{K}]_{1} & \Box & 0 & \Box & 0 \\ \Box & \Box & \Box & \Box & 0 \\ 0 & \Box & [\mathbf{K}]_{i} & \Box & 0 \\ \Box & \Box & \Box & \Box & \Box \\ 0 & \Box & 0 & \Box & [\mathbf{K}]_{p} \end{bmatrix} \begin{pmatrix} (\Delta)_{1} \\ \Box \\ (\Delta)_{i} \\ \Box \\ (\Delta)_{p} \end{pmatrix} \rightarrow \underbrace{(\mathbf{F}) = [\overline{\mathbf{K}}](\Delta)}_{[\Delta]}$$

• Global Stiffness Equation

$$(\mathbf{P}) = [\mathbf{C}]^T (\mathbf{F}) = [\mathbf{C}]^T [\overline{\mathbf{K}}] (\Delta) = [\mathbf{C}]^T [\overline{\mathbf{K}}] [\mathbf{C}] (\mathbf{u})$$

 $(\mathbf{P}) = [\mathbf{K}](\mathbf{u})$ where $[\mathbf{K}] = [\mathbf{C}]^T [\overline{\mathbf{K}}][\mathbf{C}]$

• Direct Stiffness Method

$$[\mathbf{C}]_{i}^{T}[\mathbf{K}]_{i}[\mathbf{C}]_{i} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -[\mathbf{K}_{111}]_{i} - [\mathbf{K}_{112}]_{i} - [\mathbf{K}_{122}]_{i} - [\mathbf{K}_{222}]_{i} - [\mathbf{K}_{222}]_{i$$

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5.2.3 Example



• Equilibrium Equation

$$V^{1} = V_{1}^{L} , M^{1} = M_{1}^{L}$$

$$V^{2} = V_{1}^{R} + V_{2}^{L} , M^{2} = M_{1}^{R} + M_{2}^{L}$$

$$V^{3} = V_{2}^{R} + V_{3}^{L} , M^{3} = M_{2}^{R} + M_{3}^{L}$$

$$V^{4} = V_{3}^{R} , M^{4} = M_{3}^{R}$$

• Compatibility Condition

$$w_1^L = w^1, \theta_1^L = \theta^1, w_1^R = w^2, \theta_1^R = \theta^2$$
$$w_2^L = w^2, \theta_2^L = \theta^2, w_2^R = w^3, \theta_2^R = \theta^3$$
$$w_3^L = w^3, \theta_3^L = \theta^3, w_3^R = w^4, \theta_3^R = \theta^4$$

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• Unassembled Member Stiffness Matrix

$$(\mathbf{F}) = \begin{pmatrix} (\mathbf{F})_1 \\ (\mathbf{F})_2 \\ (\mathbf{F})_3 \end{pmatrix} = \begin{bmatrix} [\mathbf{K}]_1 & 0 & 0 \\ 0 & [\mathbf{K}]_2 & 0 \\ 0 & 0 & [\mathbf{K}]_3 \end{bmatrix} \begin{pmatrix} (\Delta)_1 \\ (\Delta)_2 \\ (\Delta)_3 \end{pmatrix} = [\overline{\mathbf{K}}](\Delta)$$

• Global Stiffness Equation

$$(\mathbf{P}) = [\mathbf{E}](\mathbf{F}) = [\mathbf{E}][\overline{\mathbf{K}}](\mathbf{u}) = [\mathbf{C}]^{T}[\overline{\mathbf{K}}][\mathbf{C}](\mathbf{u}) = [\mathbf{K}](\mathbf{u})$$
$$(\mathbf{P}) = \begin{bmatrix} [\mathbf{C}]_{1}^{T} & [\mathbf{C}]_{2}^{T} & [\mathbf{C}]_{3}^{T} \end{bmatrix} \begin{bmatrix} [\mathbf{K}]_{1} & 0 & 0 \\ 0 & [\mathbf{K}]_{2} & 0 \\ 0 & 0 & [\mathbf{K}]_{3} \end{bmatrix} \begin{bmatrix} [\mathbf{C}]_{1} \\ [\mathbf{C}]_{2} \\ [\mathbf{C}]_{3} \end{bmatrix} (\mathbf{u})$$
$$= [\mathbf{K}](\mathbf{u}) = ([\mathbf{C}]_{1}^{T}[\mathbf{K}]_{1}[\mathbf{C}]_{1} + [\mathbf{C}]_{2}^{T}[\mathbf{K}]_{2}[\mathbf{C}]_{2} + [\mathbf{C}]_{3}^{T}[\mathbf{K}]_{3}[\mathbf{C}]_{3})(\mathbf{u})$$

- Member 1

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- Member 3

• Global Stiffness Matrix

$$E \begin{bmatrix} \frac{12I_1}{L_1^3} & \frac{6I_1}{L_1^2} & -\frac{12I_1}{L_1^3} & \frac{6I_1}{L_1^2} & 0 & 0 & 0 & 0 \\ \frac{6I_1}{L_1^2} & \frac{4I_1}{L_1} & -\frac{6I_1}{L_1^2} & \frac{2I_1}{L_1} & 0 & 0 & 0 & 0 \\ -\frac{12I_1}{L_1^3} & -\frac{6I_1}{L_1^2} & \frac{12I_1}{L_1^3} + \frac{12I_2}{L_2^3} & -\frac{6I_1}{L_1^2} + \frac{6I_2}{L_2^2} & -\frac{12I_2}{L_2^3} & \frac{6I_2}{L_2^2} & 0 & 0 \\ \frac{6I_1}{L_1^2} & \frac{2I_1}{L_1} & -\frac{6I_1}{L_1^2} + \frac{6I_2}{L_2^2} & \frac{4I_1}{L_1} + \frac{4I_2}{L_2} & -\frac{6I_2}{L_2^2} & \frac{2I_2}{L_2} & 0 & 0 \\ 0 & 0 & -\frac{12I_2}{L_2^3} & -\frac{6I_2}{L_2^2} & \frac{12I_2}{L_2^3} + \frac{12I_3}{L_3^3} & -\frac{6I_2}{L_2^2} + \frac{6I_3}{L_3^3} & -\frac{12I_3}{L_3^3} & \frac{6I_3}{L_3^2} \\ 0 & 0 & \frac{6I_2}{L_2^2} & \frac{2I_2}{L_2} & -\frac{6I_2}{L_2^2} + \frac{6I_3}{L_3^3} & -\frac{6I_3}{L_3^2} + \frac{4I_3}{L_3} & -\frac{6I_3}{L_3^3} & \frac{2I_3}{L_3^3} & -\frac{6I_3}{L_3^2} \\ 0 & 0 & 0 & 0 & 0 & -\frac{12I_3}{L_3^3} & -\frac{6I_3}{L_3^2} & \frac{2I_3}{L_3} & -\frac{6I_3}{L_3^2} & \frac{12I_3}{L_3^3} & -\frac{6I_3}{L_3^2} & \frac{12I_3}{L_3} & -\frac{6I$$

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• Application of support Conditions



• Final Stiffness Equation

$$\begin{pmatrix} M^{2} \\ M^{3} \\ V^{4} \\ M^{4} \end{pmatrix} = E \begin{bmatrix} \frac{4I_{1}}{L_{2}} + \frac{4I_{2}}{L_{2}} & \frac{2I_{2}}{L_{2}} & 0 & 0 \\ \frac{2I_{2}}{L_{2}} & \frac{4I_{2}}{L_{2}} + \frac{4I_{3}}{L_{3}} & -\frac{6I_{3}}{L_{3}^{2}} & \frac{2I_{3}}{L_{3}} \\ 0 & -\frac{6I_{3}}{L_{3}^{2}} & \frac{12I_{3}}{L_{3}^{3}} & -\frac{6I_{3}}{L_{3}^{2}} \\ 0 & \frac{2I_{3}}{L_{3}} & -\frac{6I_{3}}{L_{3}^{2}} & \frac{4I_{3}}{L_{3}} \end{bmatrix} \begin{pmatrix} \theta^{2} \\ \theta^{3} \\ \theta^{4} \\ \theta^{4} \end{pmatrix}$$

5.3 Frame Problems

5.3.1 Member Stiffness Matrix



• Force-Displacement Relation at Member Ends

- Beam action

$$\begin{split} m^{L} &= \frac{4EI_{e}}{L_{e}} \Theta_{e}^{L} + \frac{2EI_{e}}{L_{e}} \Theta_{e}^{R} + \frac{6EI_{e}}{L_{e}^{2}} \delta_{y}^{L} - \frac{6EI_{e}}{L_{e}^{2}} \delta_{y}^{R} \\ m^{R} &= \frac{2EI_{e}}{L_{e}} \Theta_{e}^{L} + \frac{4EI_{e}}{L_{e}} \Theta_{e}^{R} + \frac{6EI_{e}}{L_{e}^{2}} \delta_{y}^{L} - \frac{6EI_{e}}{L_{e}^{2}} \delta_{y}^{R} \\ f_{y}^{L} &= \frac{M_{e}^{L} + M_{e}^{R}}{L_{e}} = \frac{6EI_{e}}{L_{e}^{2}} \Theta_{e}^{L} + \frac{6EI_{e}}{L_{e}^{2}} \Theta_{e}^{R} + \frac{12EI_{e}}{L_{e}^{3}} \delta_{y}^{L} - \frac{12EI_{e}}{L_{e}^{3}} \delta_{y}^{R} \\ f_{y}^{R} &= -\frac{M_{e}^{L} + M_{e}^{R}}{L_{e}} = -\frac{6EI_{e}}{L_{e}^{2}} \Theta_{e}^{L} - \frac{6EI_{e}}{L_{e}^{2}} \Theta_{e}^{R} - \frac{12EI_{e}}{L_{e}^{3}} \delta_{y}^{L} + \frac{12EI_{e}}{L_{e}^{3}} \delta_{y}^{L} \end{split}$$

- Truss action

$$f_x^L = -\frac{EA}{L} (\delta_x^R - \delta_x^L)$$
$$f_x^R = \frac{EA}{L} (\delta_x^R - \delta_x^L)$$
$$V_e^L = V_e^R = 0$$

• Member Stiffness Matrix

$$\begin{pmatrix} f_x^L \\ f_y^L \\ m^L \\ m^L \\ f_x^R \\ f_y^R \\ m^R \end{pmatrix} = \frac{E}{L_e} \begin{bmatrix} A_e & 0 & 0 & -A_e & 0 & 0 \\ 0 & \frac{12I_e}{L_e^2} & \frac{6I_e}{L_e} & 0 & -\frac{12I_e}{L_e^2} & \frac{6I_e}{L_e} \\ 0 & \frac{6I_e}{L_e} & 4I_e & 0 & -\frac{6I_e}{L_e} & 2I_e \\ -A_e & 0 & 0 & A_e & 0 & 0 \\ 0 & -\frac{12I_e}{L_e^2} & -\frac{6I_e}{L_e} & 0 & \frac{12I_e}{L_e^2} & -\frac{6I_e}{L_e} \\ 0 & \frac{6I_e}{L_e} & 2I_e & 0 & -\frac{6I_e}{L_e} & 4I_e \end{bmatrix} \begin{bmatrix} \delta_x^R \\ \delta_y^R \\ \theta_e^R \\ \delta_y^R \\ \theta_e^R \end{bmatrix}$$
 or $\underline{(\mathbf{f})^e = [\mathbf{k}]^e (\mathbf{\delta})^e}$

• Transformation Matrix



$$\begin{cases} V_x = \cos\theta v_x - \sin\theta v_y \\ V_y = \sin\theta v_x + \cos\theta v_y \\ M = m \end{cases} \rightarrow \begin{pmatrix} V_x \\ V_y \\ M \end{pmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} v_x \\ v_y \\ m \end{pmatrix}$$
$$(\mathbf{V}) = [\gamma]^T (\mathbf{v}) \rightarrow (\mathbf{v}) = [\gamma] (\mathbf{V})$$

• Member End Force

$$\begin{pmatrix} F_x^1 \\ F_y^1 \\ R_y^1 \\ M^1 \\ F_x^2 \\ F_y^2 \\ M^2 \end{pmatrix}^e = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} f_x^L \\ f_y^L \\ m^L \\ f_x^R \\ m^R \end{pmatrix}^e \rightarrow \underbrace{(\mathbf{F})^e = [\Gamma]^T (\mathbf{f})^e}_{m^R}$$

• Member End Displacement

$$\begin{pmatrix} \delta_{x}^{L} \\ \delta_{y}^{L} \\ \theta_{e}^{R} \\ \delta_{x}^{R} \\ \delta_{y}^{R} \\ \theta_{e}^{R} \\ \theta_{e}^{R} \end{pmatrix}^{e} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & \sin\theta & 0 \\ 0 & 0 & 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} \Delta_{x}^{1} \\ \Delta_{y}^{1} \\ \theta_{x}^{2} \\ \Delta_{y}^{2} \\ \theta_{z}^{2} \end{pmatrix}^{e} \rightarrow \underline{(\delta)^{e} = [\Gamma](\Delta)^{e}}$$

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Member Stiffness Matrix in Global Coordinate

• Nodal Equilibrium & Compatibility

The same as the truss problems.

• Global Stiffness Matrix

$$(\mathbf{P}) = [\mathbf{C}]^T (\mathbf{F}) = [\mathbf{C}]^T [\overline{\mathbf{K}}] (\Delta) = [\mathbf{C}]^T [\overline{\mathbf{K}}] [\mathbf{C}] (\mathbf{u})$$
$$(\mathbf{P}) = [\mathbf{K}] (\mathbf{u}) \quad \text{where} \quad [\mathbf{K}] = [\mathbf{C}]^T [\overline{\mathbf{K}}] [\mathbf{C}]$$

• Direct Stiffness Method

The same as the truss problems.

5.4 Buckling of Beams and Frames

• Homogeneous Solution of Beam



• Total Potential Energy

$$\Pi = \sum_{i=1}^{p} \frac{1}{2} \int_{0}^{l} \frac{d^{2} w_{i}}{dx^{2}} EI \frac{d^{2} w_{i}}{dx^{2}} dx - \sum_{i=1}^{p} \frac{1}{2} Q_{0}^{l} \frac{dw_{i}}{dx} \frac{dw_{i}}{dx} dx - \sum_{i=1}^{p} \int_{0}^{l} w_{i} q_{i} dx$$

$$= \sum_{i=1}^{p} \frac{1}{2} \int_{0}^{l} (\frac{d^{2} w_{i}}{dx^{2}})^{T} EI \frac{d^{2} w_{i}}{dx^{2}} dx - \sum_{i=1}^{p} \frac{1}{2} Q_{0}^{l} (\frac{dw_{i}}{dx})^{T} \frac{dw_{i}}{dx} dx - \sum_{i=1}^{p} \int_{0}^{l} (w_{i})^{T} q_{i} dx$$

$$= \sum_{i=1}^{p} \frac{1}{2} (\Delta_{i})^{T} \int_{0}^{l} (\frac{d^{2} \mathbf{N}}{dx^{2}})^{T} EI \frac{d^{2} \mathbf{N}}{dx^{2}} dx (\Delta_{i}) - \sum_{i=1}^{p} \frac{1}{2} Q(\Delta_{i})^{T} \int_{0}^{l} (\frac{d\mathbf{N}}{dx})^{T} \frac{d\mathbf{N}}{dx} dx (\Delta_{i}) - \sum_{i=1}^{p} (\Delta_{i})^{T} \int_{0}^{l} (\mathbf{N})^{T} q_{i} dx$$

$$= \frac{1}{2} \sum_{i=1}^{p} (\Delta_{i})^{T} ([\mathbf{K}_{i}^{0}] - [\mathbf{K}_{i}^{G}]) (\Delta_{i}) - \sum_{i=1}^{p} (\Delta_{i})^{T} (\mathbf{f}_{i})$$

$$= \frac{1}{2} (\mathbf{u})^{T} \sum_{i=1}^{p} [\mathbf{C}_{i}]^{T} ([\mathbf{K}_{i}^{0}] - [\mathbf{K}_{i}^{G}]) [\mathbf{C}_{i}] (\mathbf{u}) - \sum_{i=1}^{p} [\mathbf{C}_{i}]^{T} (\mathbf{f}_{i})$$

$$= \frac{1}{2} (\mathbf{u})^{T} \sum_{i=1}^{p} [\mathbf{C}_{i}]^{T} [\mathbf{K}_{i}] [\mathbf{C}_{i}] (\mathbf{u}) - (\mathbf{u})^{T} \sum_{i=1}^{p} [\mathbf{C}_{i}]^{T} (\mathbf{f}_{i})$$

$$= \frac{1}{2} (\mathbf{u})^{T} \sum_{i=1}^{p} [\mathbf{C}_{i}]^{T} [\mathbf{K}_{i}] [\mathbf{C}_{i}] (\mathbf{u}) - (\mathbf{u})^{T} \sum_{i=1}^{p} [\mathbf{C}_{i}]^{T} (\mathbf{f}_{i})$$

$$\mathbf{K}_{i}^{0} = \frac{EI_{i}}{L_{i}} \begin{bmatrix} \frac{12}{L_{i}^{2}} & \frac{0}{L_{i}} & -\frac{12}{L_{i}^{2}} & \frac{0}{L_{i}} \\ \frac{6}{L_{i}} & 4 & -\frac{6}{L_{i}} & 2 \\ -\frac{12}{L_{i}^{2}} & -\frac{6}{L_{i}} & \frac{12}{L_{i}^{2}} & -\frac{6}{L_{i}} \end{bmatrix}, \quad \mathbf{K}_{i}^{G} = Q \begin{bmatrix} \frac{6}{5L_{i}} & \frac{1}{10} & -\frac{6}{5L_{i}} & \frac{1}{10} \\ \frac{1}{10} & \frac{2L_{i}}{15} & \frac{1}{10} & -\frac{L_{i}}{30} \\ -\frac{6}{5L_{i}} & \frac{1}{10} & \frac{6}{5L_{i}} & -\frac{1}{10} \\ \frac{6}{5L_{i}} & \frac{1}{10} & \frac{6}{5L_{i}} & -\frac{1}{10} \\ \frac{1}{10} & -\frac{L_{i}}{30} & -\frac{1}{10} \\ \frac{1}{10} & -\frac{L_{i}}{30} & -\frac{1}{10} & \frac{2L_{i}}{15} \end{bmatrix}$$

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• Principle of Minimum Potential Energy for $Q < Q_{cr}$

$$\Pi = \frac{1}{2} (\mathbf{u})^{T} [\mathbf{K}] (\mathbf{u}) - (\mathbf{u})^{T} (\mathbf{P}) = \frac{1}{2} \sum_{i=1}^{n} u_{i} \sum_{j=1}^{n} K_{ij} u_{j} - \sum_{i=1}^{n} u_{i} P_{i}$$

$$\frac{\partial \Pi}{\partial u_{k}} = \frac{1}{2} \sum_{i=1}^{n} \frac{\partial u_{i}}{\partial u_{k}} \sum_{j=1}^{n} K_{ij} u_{j} + \frac{1}{2} \sum_{i=1}^{n} u_{i} \sum_{j=1}^{n} K_{ij} \frac{\partial u_{j}}{\partial u_{k}} - \sum_{i=1}^{n} u_{i} P_{i}$$

$$= \frac{1}{2} \sum_{j=1}^{n} K_{kj} u_{j} + \frac{1}{2} \sum_{i=1}^{n} u_{i} K_{ik} - P_{k} = \frac{1}{2} \sum_{j=1}^{n} K_{kj} u_{j} + \frac{1}{2} \sum_{i=1}^{n} K_{kj} u_{j} - P_{k} = \frac{1}{2} \sum_{j=1}^{n} K_{kj} u_{j} - P_{k} = 0 \text{ for } k = 1, \dots, n \rightarrow [\mathbf{K}] (\mathbf{u}) = (\mathbf{P})$$

• Calculation of the Critical Load

$$Det([\mathbf{K}]) = Det([\mathbf{K}^0] - [\mathbf{K}^G]) = 0$$

• Frame Members

$$\begin{pmatrix} f_x^L \\ f_y^L \\ m^L \\ m^L \\ f_y^R \\ m^R \end{pmatrix} = \left(\frac{E}{L_e} \begin{pmatrix} A_e & 0 & 0 & A_e & 0 & 0 \\ 0 & \frac{12I_e}{L_e^2} & \frac{6I_e}{L_e} & 0 & -\frac{12I_e}{L_e^2} & \frac{6I_e}{L_e} \\ 0 & \frac{6I_e}{L_e} & 4I_e & 0 & -\frac{6I_e}{L_e} & 2I_e \\ A_e & 0 & 0 & A_e & 0 & 0 \\ 0 & -\frac{12I_e}{L_e^2} & -\frac{6I_e}{L_e} & 0 & \frac{12I_e}{L_e^2} & -\frac{6I_e}{L_e} \\ 0 & \frac{6I_e}{L_e} & 2I_e & 0 & -\frac{6I_e}{L_e^2} & -\frac{6I_e}{L_e} \\ 0 & \frac{6I_e}{L_e} & 2I_e & 0 & -\frac{6I_e}{L_e^2} & 4I_e \\ \end{pmatrix} - Q \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{6}{5L_e} & \frac{1}{10} & 0 & -\frac{6}{5L_e} & \frac{1}{10} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{6}{5L_e} & \frac{1}{10} & 0 & \frac{6}{5L_e} & -\frac{1}{10} \\ 0 & \frac{1}{10} & -\frac{L_e}{30} & 0 & -\frac{1}{10} & \frac{2L_e}{15} \\ \end{pmatrix} \begin{pmatrix} \delta_x^L \\ \delta_y^L \\ \theta_e^L \\ \theta_e^L \\ \delta_y^R \\ \theta_e^R \end{pmatrix}$$

$$P = EA \frac{\delta_x^L - \delta_x^R}{L_e}$$

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